Abstract state machines as a tool for history of logic

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Ibn Sīnā (Avicenna), Uzbekhistan and Persia 980–1037

Ibn Sīnā's major work in Logic was the logic section of *Shifa'* ('Cure'), over 2000 pages of Arabic.

In the volume *Qiyās* ('Syllogism') he explains how to formalise texts as sequences of syllogisms, and how to fill gaps where premises are missing.

In one place he suddenly gives, with virtually no explanation, 64 examples for the reader.

What were these examples supposed to teach?

Samples:

> [Problem 1.] Suppose the goal is universally quantified affirmative, namely 'Every *C* is an *A*', and suppose that the found premises are 'Every *C* is a *B*' and 'Every *D* is an *A*'. Then if it's clear that 'Every *B* is a *D*', your syllogism is in good order; otherwise it needs a middle.
>
> [Problem 4, goal 'No *C* is an *A*'.] Suppose the found [premises] are 'No *C* is a *B*' and 'Every *D* is an *A*'. Then it can't be used.
>
> [Problem 37.] If the goal is universally quantified affirmative [thus: 'Every *C* is an *A*']; and you have [the premises] 'Every *D* is a *B*' and 'Every *B* is an *A*', and 'Every *C* is a *D*' is attached, this makes [the syllogism] determinate.
>
> Work through the remaining cases of this kind for yourself.

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After staring at the list of examples, I conjectured:

- Ibn Sīnā is describing an algorithm which, given some premises known to be true, and a conclusion to be derived,
- *either* declares that there is no proof of the conclusion using the given premises irredundantly,
- *or* lists all possible ways of adding premises so as to prove the conclusion, using the given premises irredundantly

where 'irredundantly' means that without all of the given premises, the conclusion doesn't follow.

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This is a sort of proof search algorithm.

Snag. No search algorithms at all have previously been reported in Arabic mathematics of that date.

Al-Khalīl, 8th century, described some algorithms for enumerating strings, in connection with the design of dictionaries.

Al-Khwārizmi ('Mr Algorithm'), 9th century, described an algorithm for solving quadratic equations, and proved its correctness using Euclidean geometry.

So a case has to be made for this reading of Ibn Sīnā.

Method. Write an abstract state machine (ASM) describing the algorithm.
Determine how far the steps of the machine are determined by Ibn Sīnā's text.

The resulting ASM is in the YuriFest volume, to honour Yuri Gurevich's invention of ASMs.

It was very helpful to be able to call on the ASM specification of Prolog proof search by Egon Börger and Dean Rosenzweig for comparison.

The ASM is about four pages long.

A simulation of it was run on Ibn Sīnā's 64 Examples.
It gave the same answers as Ibn Sīnā in all but 5 cases.

Four of these five are almost certainly corruptions of the text — these are common.

The fifth is the result of a glitch in the standard enumeration of syllogisms (at least since Philoponus in 6th century), which in effect takes this example outside the domain of the algorithm.

Some relevant logic

There are four kinds of syllogistic sentence:

- Every A is a B.
- No A is a B.
- Some A is a B.
- Not every A is a B.

Each sentence has two terms, A and B, normally assumed to be distinct.

So given two terms A, B, there are 8 syllogistic sentences with these two terms.

If a set T of $\geqslant 3$ syllogistic sentences entails a syllogistic sentence ψ irredundantly,
then $T \cup \{\neg\psi\}$ is inconsistent and can be arranged in a circle,
where each term appears exactly twice and in adjacent sentences, and no two sentences have the same terms.
(Ibn Sīnā certainly knew this.)

Such a circle is inconsistent if and only if:

- in exactly one of the sentences, the second term occurs negatively, and
- each term has at least one negative occurrence.

(There is no evidence that Ibn Sīnā knew this.)

We abbreviate ‘irredundant inconsistent circle’ to IIC.

If an arc C is removed from an IIC,
there is an (essentially) unique weakest sentence θ_C
that can be put in the gap to form an IIC.

The two terms of θ_C can be read off from
the circle minus C.
Ibn Sīnā knew this; his algorithm involves listing the 8
sentences with these two terms.

When C is given, then θ_C can be calculated by
taking a pair of adjacent sentences in C,
deriving their syllogistic conclusion, and iterating.

Ibn Sīnā knew this and explains it in detail.
He normally starts at the leftmost pair, reading clockwise.
(In Arabic, read ‘right’ for ‘left’.)

The ASM has a module SYNTHESISE which shrinks down
arcs in this way.

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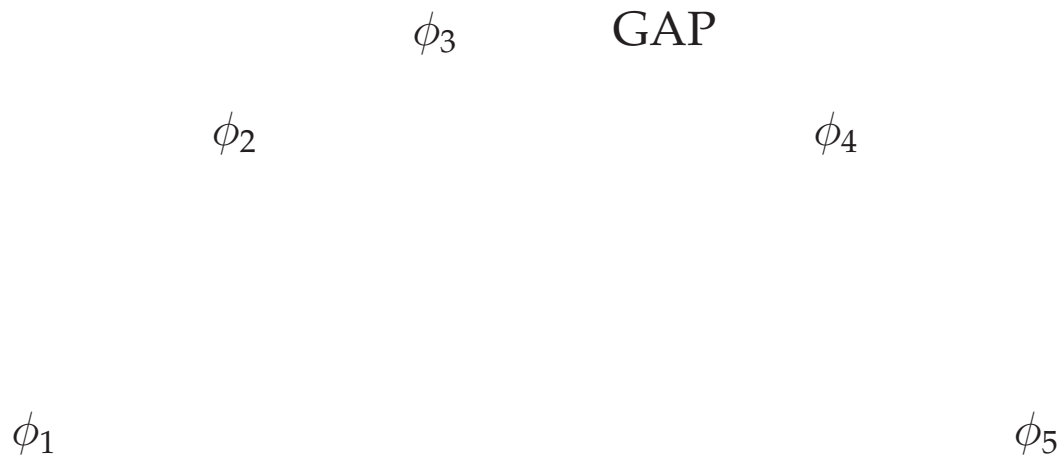

$(\neg)\psi$

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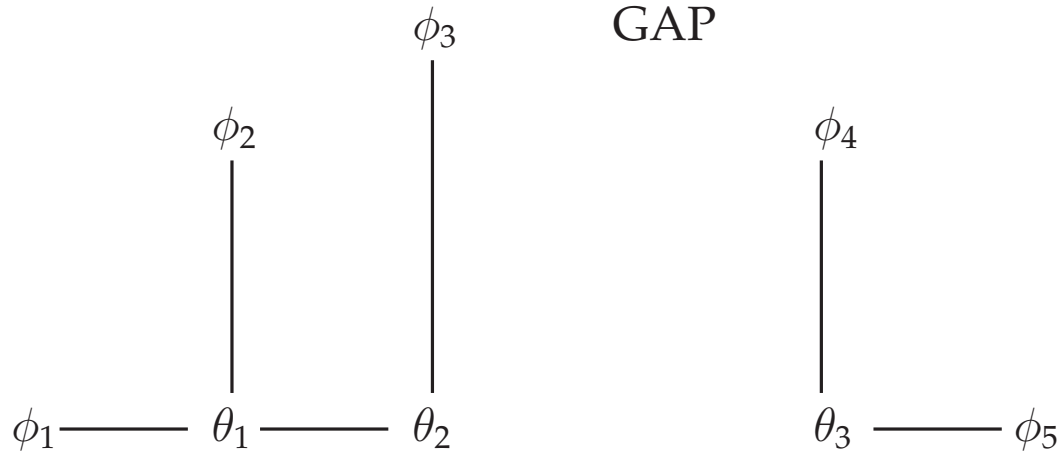


$(\neg)\psi$

If this diagram can't be constructed,
the algorithm delivers 'No solution'.
If it can, the problem reduces to a small circle:

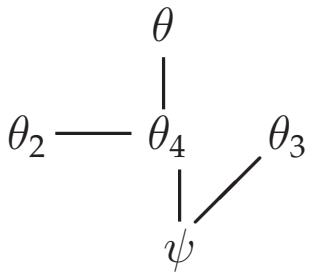
$$\text{GAP}$$

$$\theta_2 \qquad\qquad \theta_3$$

$$(\neg)\psi$$

There are 8 candidate sentences θ to fill the gap.
The algorithm takes each in turn until a suitable one is found or the list is exhausted. (Ibn Sīnā illustrates this.)

A prima facie successful θ yields by SYNTHESISE:



If θ is prima facie successful,
the algorithm asks whether θ is known to be true.

If Yes, the problem is solved.

If No, ‘it needs a middle’, i.e. we must find $\chi_1, \ldots, \chi_n$ which entail θ and are known to be true.

Ibn Sīnā’s examples suggest he started with χ_1, which has one term in common with θ_2 and one new term (the ‘middle’).

$$\begin{array}{ccccc} & \text{GAP} & & & \\ \theta_2 & & \theta_3 & \Rightarrow & \theta_2 \\ & (\neg)\psi & & & \end{array} \qquad \begin{array}{ccc} & \chi_1 \quad \text{GAP} & \\ & & \theta_3 \\ & (\neg)\psi & \end{array}$$

So there is a new gap between χ_1 and θ_3.

Here we would recurse, and backtrack if a new choice of χ_1 is needed. What does Ibn Sīnā do?

Differences from modern proof search

1. The sentences already known to be true are not given explicitly.
Ibn Sīnā assumes only that, given a sentence,
we know whether or not it is known to be true.
So this information is best given by a dedicated function.

2. At Ibn Sīnā's date, neither recursion nor backtracking appears in any Arabic algorithm.
In our ASM, we allow a splitting into many agents,
each of which handles one possibility for χ_1.
So the ASM is multi-agent.

3. Ibn Sīnā believed that the middle can't be found algorithmically.

When obvious heuristics fail, he falls back on prayer, alcohol and sleep.
These allow the Active Intellect
(essentially the same thing as Gödel's 'the a priori')
to implant suggestions in our minds.

Our multi-agent ASM incorporates the Active Intellect as a global agent.

Since Ibn Sīnā has really only one kind of calculation, namely deriving a consequence from two syllogistic sentences,
the main job of the algorithm is to control the order in which these derivations are made.

This was Ibn Sīnā's own view too:

> 'When you put the steps in this order, as I have shown you, you will reach the required terms and syllogistic derivations.'

Tantalising remark

In all of Ibn Sīnā's examples there is a single gap.
He notes that we can pose the same problem with two separate gaps in the circle.
He says this is more complicated and he will deal with it in the 'Appendices'.

It's not known that Ibn Sīnā ever wrote these Appendices.
But he would have needed to interweave suggested fillings for the two gaps.
This would have needed some backtracking device.

Some verdicts on this use of ASMs

- The ability to choose any appropriate level of detail, a key property of ASMs, was essential.
- The construction of the ASM constantly raised questions to address to Ibn Sīnā's text.
  Very often there were answers.
  Sometimes they were in other parts of Ibn Sīnā's text; this helped to show the interdependence of sections of Ibn Sīnā's book.

- ASMs are descriptive devices.
  Our ASM itself describes very precisely the things about which Ibn Sīnā had clear intentions.
  ASMs are never vague,
  so it can't represent what he was vague about.
  Nevertheless it was a great help in identifying where he was vague.

"I'm in touch with a very special intelligence.
This intelligence I find nourishing.
I have been nourished by it.
It's enlarged me."

Harold Pinter, "No Man's Land"

Yuri: do you remember watching this play in 1993
with Zoe and Helen and me?

The very special intelligence is yours,
and we have all been enlarged by it.