

## What would count as Ibn Sīnā (11th century Persia) having first order logic?

Wilfrid Hodges  
Herons Brook, Sticklepath, Okehampton  
March 2012

<http://wilfridhodges.co.uk>



Ibn Sina, 980–1037

Ibn Sīnā claimed that Euclid's *Elements* is 'syllogistic', which should mean that all of Euclid's arguments can be formalised as valid syllogisms.

The implication is that Ibn Sīnā knew how to validate all the arguments in the *Elements*.

For this he would have needed most (perhaps all) of full first-order logic.

Did he have this?

Ibn Sīnā wrote masses about logic.

I have in preliminary translation the equivalent of about 2,200 pages of his Arabic writings on logic.

So we know pretty well what his views were, and what he was aware of.

But all this information doesn't answer the question:

*How should one describe Ibn Sīnā's logical expertise from the point of view of a modern logician?*

The question is methodological rather than historical.

5

### 1. What would Ibn Sīnā himself have counted as validating the arguments in *Elements*?

We can answer this, though you won't find the answer in standard histories of logic.

We can draw out the answer by examining how Ibn Sīnā validates an argument in the proof of Proposition 1 of *Elements* (trans. Heath):

Each of the straight lines  $CA, CB$  is equal to  $AB$ .  
And things which are equal to the same thing are also equal to one another;  
therefore  $CA$  is also equal to  $CB$ .

I use some notation that I'll explain as we go.

6

$$\begin{array}{l} \frac{\{C \equiv B\} \quad \{B \equiv D\}}{\{C \equiv B\} \text{ and } \{B \equiv D\}} \quad (\alpha) \text{ (not mentioned by Ibn Sīnā)} \\ \quad \quad \quad | \quad (\beta) \\ \frac{\{B\} \{\text{has } C \equiv \text{it and is } \equiv D\}}{\text{Some } \{line\} \{\text{has } C \equiv \text{it and is } \equiv D\}} \quad (\gamma) \\ \quad \quad \quad | \quad (\delta) \\ \frac{\{C, D\} \text{ is a } \{\text{pair of lines with some line } \equiv\text{-between them}\} \quad \text{Every } \{\text{pair of lines with some line } \equiv\text{-between them}\} \text{ is a } \{\text{pair of } \equiv \text{lines}\}}{\{C, D\} \text{ is a } \{\text{pair of } \equiv \text{lines}\}} \quad (\epsilon) \\ \quad \quad \quad | \quad (\zeta) \\ \quad \quad \quad C \equiv D \end{array}$$

7

### 1.1 Local formalising

Ibn Sīnā regarded a complex argument as a tree of inference steps, not a tree of propositions.

In fact he distinguished between a proposition as the conclusion of a step and the same proposition as premise of the next step.

Logic is used to validate *isolated inference steps*.  
(Just as in grammar we parse each sentence separately.)  
This was universal practice before Peano and Frege.

8

Ibn Sīnā claimed in his Autobiography that he had trained himself by working through all the major works of philosophy and extracting the syllogistic inference steps, each of which he filed separately. He seems to have included Euclid's *Elements*.

He claimed also that for each syllogistic step he did two further things:

- ▶ he determined the terms, and
- ▶ he took care of the conditions.

We will see what these moves consisted of.

## 1.2 Logic validates only two-premise inference steps

Ibn Sīnā justifies this as follows. The main logical inference procedure is to take two premises and discover a descriptive term which occurs in both (cf. Unification). This term is then eliminated (cf. Resolution), and the remaining pieces of the premises are recombined into a new proposition which expresses new knowledge.

One-premise inferences don't allow this unification and recombination, basically because the conclusion contains no information beyond what was in the premise.

At first approximation, 'syllogism' for Ibn Sīnā means inference step that provides new information in the way described above.

The diagram above contains one syllogism, namely ( $\varepsilon$ ). Step ( $\gamma$ ), which Ibn Sīnā mentions explicitly, is an example of an 'entailment' (*luzūm*) but not a syllogism. Probably the same holds for step ( $\alpha$ ), which he doesn't mention.

Ibn Sīnā emphasises that his notion of syllogism includes some argument forms not considered by Aristotle, for example some propositional arguments.

## 1.3 Determining the terms

A logician validates syllogistic inferences by finding terms that appear either explicitly or implicitly in the premises and conclusion, and that work as descriptive terms for the syllogistic procedure described above.

After finding the terms, the logician checks that they are arranged in one of the logically accepted patterns or *moods* — which Ibn Sīnā expects his students to memorise.

The mood for ( $\varepsilon$ ) above is 'Barbara with singular minor':

$X$  is a  $Y$ . Every  $Y$  is a  $Z$ . Therefore  $X$  is a  $Z$ .

To indicate the terms, Aristotle drew up a table as we do today, listing the terms and labelling each with a letter. This procedure dropped out after Aristotle and was revived by Boole in 1854 under the influence of Peacock's 'symbolical algebra'.

Between Aristotle and Boole, logicians indicated the terms by paraphrasing into a normal form. These normal forms depended on the language, and Ibn Sīnā's form used the fact that Arabic (unlike English) is a 'topic-prominent language'.

In the diagram I replace his convention by curly brackets to pick out the terms.

Steps ( $\beta$ ), ( $\delta$ ) and ( $\zeta$ ) in our diagram are paraphrases to allow us to indicate the terms used in the relevant inference steps.

Leibniz described such steps as ‘linguistic analyses’, and Frege condemned them as ‘changes of viewpoint’ which are not under the control of logic.

Note that paraphrase ( $\delta$ ) combines two items into a pair. This device was introduced by Alexander of Aphrodisias in the late 2nd century AD in answer to a question of Galen about how to handle relations in arithmetic.

## 2. Modern critique of Ibn Sīnā’s requirements for validating

We noted one major criticism from Frege: the paraphrases are not under logical control.

According to Frege’s analysis, the paraphrases appeared because the old inference rules were too closely linked to the surface syntax of natural languages.

E.g. paraphrase ( $\beta$ ) is needed because the syllogistic rules apply to single occurrences of terms, and the term  $B$  occurs twice.

For further discussion see Frege’s introduction to *Begriffsschrift*, which is still fresh.

A second critique, which became popular among some logicians and historians of logic in around 1970, is that Aristotelian logic is unable to handle multiple quantification.

This can’t be right, because no standard calculus for full first order logic contains an inference rule that applies to more than one quantifier.

A third critique has more force.

Aristotelian logic has no means of applying inference rules below the top syntactic level of the premises.

Its only weapon against this restriction is to paraphrase the premises so that syntactically deeper material comes to the top level.

Leibniz worked on this problem but was held up by the limitations of paraphrase.

Frege made several innovations to deal with this problem. One relevant to Ibn Sīnā is a propositional axiom in *Begriffsschrift*:

$$((c \rightarrow (b \rightarrow a)) \rightarrow ((c \rightarrow b) \rightarrow (c \rightarrow a))).$$

Given modus ponens, this axiom validates the sequent

$$(c \rightarrow (b \rightarrow a)), (c \rightarrow b) \vdash (c \rightarrow a).$$

This sequent is the result of applying

$$(b \rightarrow a), b \vdash a$$

inside conditionals ( $c \rightarrow \star$ ).

In a natural deduction calculus we get the same effect by *assuming*  $c$ , then applying modus ponens several times, then *discharging* the assumption by  $\rightarrow$ I.

Frege took the view that this machinery of assuming and discharging is adopted for 'stylistic reasons', in order to avoid propositions which have many conditions and hence 'eine ungeheuerliche Länge'. But in his view, these propositions are needed for a correct explanation of the underlying argument.

### 3. Did Ibn Sīnā have an answer to this third critique?

At first sight no. Ibn Sīnā constantly emphasises that syllogistic rules treat the descriptive terms of a proposition as black boxes. The internal structure of these terms is invisible for purposes of logic.

Nevertheless it seems that Ibn Sīnā comes to Frege's position, though by an indirect and rather astonishing route.

The clue is to take seriously Ibn Sīnā's remarks about 'taking care of the conditions'. (I think nobody has done so until now. Gutas stated that the conditions are about modalities.)

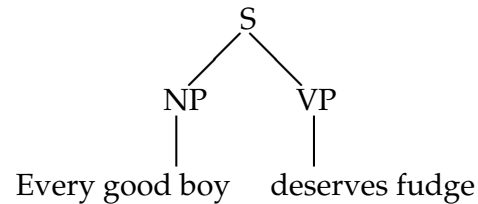
Ibn Sīnā bases his logic on the analysis of sentences. His analysis agrees with what you find in the early chapters of any textbook of generative grammar.

A crucial difference is that Ibn Sīnā is describing not syntax but meanings.

He believes (and says) that the syntax of any natural language reflects the way that the meanings of the words are composed to reach the meaning of the sentence.

Also syntax contains rules for putting the words in linear order. These are needed because the structure of meanings is (he says) not one-dimensional.

Thus each basic sentence analyses into two components, Noun Phrase and Verb Phrase:



Ibn Sīnā calls the NP 'subject', and he calls the VP 'predicate'.

In modern accounts the internal structure of the NP and the VP is described by X-bar theory.

By this theory the NP contains a noun, its 'head', and the remaining parts of the NP are 'adjoined to' or 'adjuncts of' this head.

Likewise the VP, except that its head is a verb.

In the example, the head of the NP is 'boy' and the head of the VP is 'deserves'.

Ibn Sīnā calls the heads 'thing' (*šay*), and he calls the adjuncts 'addition' (*ziyāda*) or 'adjunct' (*lāḥiq*) or 'condition' (*šarṭ*).

Ibn Sīnā believes that any syllogistic inference with this sentence as premise involves four items:

- ▶ the two heads,
- ▶ the quantifier adjunct on the NP head, and
- ▶ the presence or absence of negation in the copula joining NP to VP.

After establishing that a syllogistic inference holds on these items, the logician should go back to the adjuncts (i.e. 'take care of the conditions') and check whether they damage the inference. The default is that they do not.

This approach seems amazingly cack-handed.

But strangely it works, and gives Ibn Sīnā a logical rule that is sound and far stronger than anything in the literature before the 19th century.

We can formalise his default assumption and write down conditions under which it is completely sound.

The formalism uses notions of variable binding etc. that were unknown to Ibn Sīnā, but I think it clearly catches part of his intuition.

**Ibn Sīnā's Rule** (in standard first-order logic):

Let  $T$  be a set of formulas and  $\phi, \psi$  formulas.  
 Let  $\delta(p)$  be a formula in which  $p$  occurs only positively, and  $p$  is not in the scope of any quantifier on a variable free in some formula of  $T$ . Suppose

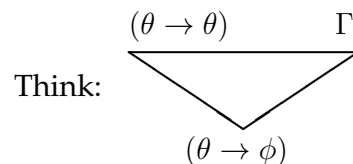
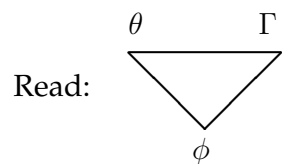
$$T, \phi \vdash \psi.$$

Then

$$T, \delta(\phi) \vdash \delta(\psi).$$



He concludes: Throughout the relevant section of the argument, each proposition should be understood as beginning with an implicit 'If  $\theta$ ', so for purposes of logical validation we should make this clause explicit.



The  $(\theta \rightarrow \theta)$  at right top is an axiom and can be discarded.



Ibn Sīnā himself applies this rule as follows, to explain the logic of making and then discharging assumptions.

He notes that when mathematicians state an assumption, they normally first introduce it not with 'Suppose  $\theta$ ' but with 'If  $\theta$ '.

(I checked this in the Arabic text of *Elements* Bk I, and he is right.)

He then notes that these writers don't repeat the assumption  $\theta$  before their final statement, even when they draw conclusions that depend on it. (The position in *Elements* is more complicated than he allows, but his view is at least plausible.)



Ibn Sīnā's Rule is not a conventional rule of inference, and it can't be written as a sequent.

Instead it is a rule for generating new inference rules from old ones.

Ibn Sīnā uses it as an implicit heuristic, not an explicit rule.

He does explicitly list several special cases of it, and claims to prove some of them.



#### 4. So did Ibn Sīnā have full first order logic?

I give two opposite answers, both true.

#### The answer no

Ibn Sīnā's Rule as stated uses the notion of a positive occurrence.

Ibn Sīnā shows no knowledge of any such notion (or of related notions like 'distributed').

One could argue that if  $\phi$  has a 'condition' added to it, then in the resulting compound,  $\phi$  will occur positively.

For example  $(\phi \rightarrow \theta)$  doesn't express adding a 'condition' to  $\phi$ .

#### The answer yes

If we collect up into a single formal system

- ▶ the syllogistic moods that Ibn Sīnā states,
- ▶ some non-syllogistic inferences that he clearly recognises,
- ▶ Ibn Sīnā's Rule and
- ▶ some structural rules that are needed for handling Ibn Sīnā's Rule

we can get a sound and complete first order calculus. It seems to be new.

Unfortunately Ibn Sīnā himself didn't see this.

He even claimed to prove special cases of his Rule where  $p$  occurs negatively in  $\delta(p)$ .

(The proofs are garbage, if only because the copiers couldn't follow them and got the letters muddled.)

Further examination reveals that Ibn Sīnā had no notion of the scope of a negation, and made mistakes when this notion is needed.



Yet further examination shows:

- ▶ that Ibn Sīnā had no notion of the scope of a quantifier either, and
- ▶ that in both cases he knew (and said) that he was missing something crucial, but he was prevented from finding it by his insistence that compound meanings are not linearly ordered.

This is not the only case where earlier thinkers missed things that we see, not because they were blind, but because they saw something else that we fail to notice.