LINDSTRÖM LECTURE 2013

Ibn Sīnā on discharging assumptions in proofs

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http://wilfridhodges.co.uk/arabic36.pdf

Ibn Sina, 980–1037

The 'usual way' of writing reductio ad absurdum, according to Ibn Sīnā:

$$\dfrac{\dfrac{\dfrac{\text{Not not every } C \text{ is a } B}{\text{Every } C \text{ is a } B} \qquad \text{Every } B \text{ is an } A}{\text{Every } C \text{ is an } A} \qquad \text{Not every } C \text{ is an } A}{\dfrac{\perp}{\text{Not every } C \text{ is a } B}}$$

He gives this only as an example. But he knows that the part under the assumption could be more elaborate.
So a more general version of his 'usual way' is:

$$\dfrac{\neg\neg\phi}{\phi} \qquad \Psi$$

$$\dfrac{\chi \qquad \neg\chi}{\dfrac{\perp}{\neg\phi}}$$

For Ibn Sīnā, a justification of this argument is a demonstration that the conclusion can be reached from the premises (except $\neg\neg\phi$) by steps that are all intuitively convincing.

To avoid circular procedures, the demonstration must introduce no new concepts.

In particular Ibn Sīnā would not accept a metatheoretic justification, for example using model theory or a theory of dialogues.

He achieves his justification as follows.

He observes that in practice we often introduce an assumption with 'If . . . ', and then keep using the assumption without repeating it.

This observation would certainly be correct if the assumption was introduced with 'Suppose' or 'Let' (*li-takun* in Arabic).
But in fact it is also true with 'If' (*'in* or *law* in Arabic).

We illustrate from the Arabic translation of Euclid *Elements* i Prop. 27:

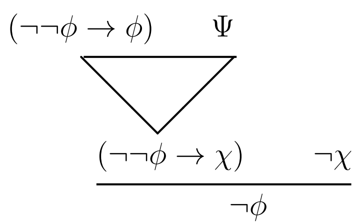
"When a straight line lies across two straight lines so that the two symmetrically-opposite angles are equal then the two lines are parallel.
. . .

Demonstration: **If the two are not parallel** then when they are both extended on one of the two sides, they meet. So we extend them on the side *BD* so they meet in a point *K* if that is possible, so the angle *AHT* external to the triangle *KTH* is greater than the internal angle *KTH*, as was proved in the demonstration of 16 of i, and this is absurd."

Therefore Ibn Sīnā concludes that the assumption

'If $\neg\neg\phi$'

is meant throughout the lefhand side of the proof, until the contradiction is reached. Thus:



In other words:

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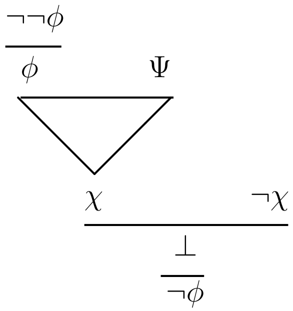

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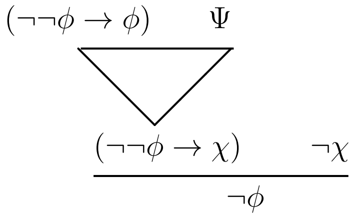

Though Ibn Sīnā doesn't state it, the same analysis would work for any proof by $\to$-introduction:

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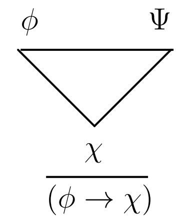

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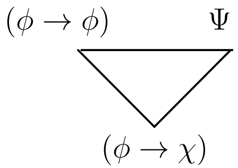


No surprise that he didn't mention this.
The notion of discharging an assumption was formulated only centuries later (19th century??).

We have to check that every step of the 'thought' argument really is intuitively convincing.

At the top we have $(\neg\neg\phi \to \phi)$, which Ibn Sīnā regards as an obvious axiom.

At the bottom we have modus tollens

$$\frac{(\neg\neg\phi \to \chi) \qquad \neg\chi}{\neg\phi}$$

which he discusses at length elsewhere.

It remains to show that if an inference

$$\theta, \psi \ \vdash \ \chi$$

is intuitively convincing, then so is

$$(\phi \to \theta), \psi \ \vdash \ (\phi \to \chi).$$

This is an interesting question, worth generalising. Suppose $\delta(p)$ is a formula containing the propositional variable p. Then when does $\theta, \psi \vdash \chi$ guarantee

$$\delta(\theta), \psi \ \vdash \ \delta(\chi)?$$

Ibn Sīnā has a section of his book *Qiyās* devoted to showing that if $\theta, \psi, \vdash \chi$ is a valid Aristotelian syllogism then this shows the validity of

$$\forall t(\phi \to \theta), \psi \ \vdash \ \forall t(\phi \to \chi)$$

(Here $\delta(p)$ is $\forall t(\phi \to p)$. He also takes $\exists t(\phi \wedge p)$.)

He gives no argument that covers all cases.
Instead he invites us to inspect each syllogism and then convince ourselves that the new argument is valid too.
This is not a proof; it's a device for convincing us by inviting us to inspect many examples.
In early work (1930) Tarski used such arguments, calling them 'empirical'.

FACT (in any standard natural deduction system):

Let Ψ be a set of formulas and θ, χ formulas.
Let $\delta(p)$ be a formula in which p occurs only positively, and p is not in the scope of any quantifier on a variable free in some formula of Ψ.
Suppose $\theta, \Psi \vdash \chi$. Then

$$\delta(\theta), \Psi \ \vdash \ \delta(\chi).$$

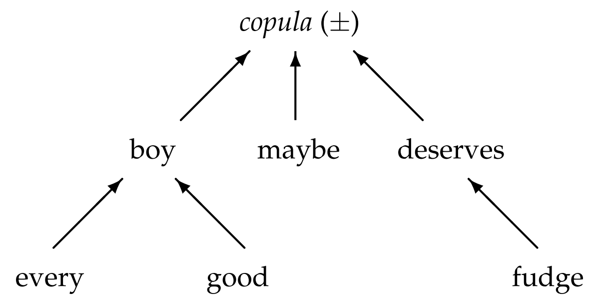
Call this *Ibn Sīnā's Principle*. It gives exactly what he needs for the preservation of validity in his treatment of assumptions.

Even if he had had the formal skills (which he didn't), Ibn Sīnā could never have given Ibn Sīnā's Principle, for two main reasons.

First, he had no notion of scope, either of quantifiers or of negations.
He knew he was missing something, when he had to handle sentences with two quantifiers of different type.

I document this in an essay in *Logic Without Borders: Essays in Honour of Jouko Väänänen*, ed. Roman Kossak et al., forthcoming.

His lack of any notion of scope causes one of the main differences between his analysis of the structure of complex meanings and Frege's *Begriffsschrift*. Recall:



Inventing the notion of scope, Frege moves 'every' up to the top where it marks out its scope.

And second, Ibn Sīnā had no notion of positive and negative occurrences.

For example he has a section trying to show that with suitable adjustments we can take

$$\delta(p) = (p \to r).$$

Thus (*Qiyās* 337.12):
Assuming it's possible that every B is an A:
Every C is a B;
and whenever every B is an A, then r.
$\vdash$ It's possible that when every C is an A, then r.

The explanation that follows in the manuscripts is garbage.
Probably the text is corrupt, but could there be any plausible result along these lines?

More generally the present evidence is that none of Ibn Sīnā's innovations in modal logic holds any water at all.

Ibn Sīnā's lack of any notion of positive and negative occurrences tallies with the fact that he has no notion of distribution, unlike the Latin Terminists.

Ibn Sīnā's Principle has two important properties in the context of the differences between Aristotelian logic and modern logic.

The first is that it applies uniformly across long stretches of argument.
Before Frege and Peano, logicians always validated single inference steps one at a time, not stretches of argument. (They used 'local formalising'.)

So they were unable to formulate operations like $\to$-introduction in proper generality, since the discharge can come arbitrarily far away from the assumption.

The second is that it allows us to apply rules at arbitrary syntactic depth within a formula,
by applying them to a stripped-down version of the formula and then putting back the stripped-off pieces.

The first explicit statement that a logical rule operates at arbitrary depth may be in Boole 1847:

"Let us represent the equation of the given Proposition under its most general form,

$$a_1 t_1 + a_2 t_2 ... + a_r t_r = 0$$

... Now the most general transformation of this equation is

$$\psi(a_1 t_1 + a_2 t_2 ... + a_r t_r) = \psi(0),$$

provided that we attribute to ψ a perfectly arbitrary character, allowing it even to involve new elective symbols, having *any proposed relation* to the original ones." (Boole's italics)

The proof calculus of Frege's *Begriffsschrift* (1879) also breaks through the depth restriction.
See how Frege does it.
In *Begriffsschrift* he has a propositional axiom

$$((r \to (b \to a)) \to ((r \to b) \to (r \to a))).$$

This can be seen as applying

$$\delta(p) = (r \to p)$$

to conclusion and both premises of

$$(b \to a), b \ \vdash \ a.$$

In the later *Grundgesetze* i §14 Frege shows that there is a derived rule going from

$$(b \to a), b \ \vdash \ a$$

to

$$\delta(b \to a), b \ \vdash \ \delta(a)$$

where

$$\delta(p) = (r_1 \to (r_2 \to \ldots \to (r_n \to p) \ldots).$$

It's plausible that he designed the *Begriffsschrift* notation to make this move highly visible, supporting exactly the intuition that Ibn Sīnā had invoked.

In fact we can confirm that Ibn Sīnā's Principle breaks through these two logjams
(local formalising and shallow proof rules)
by using it and various more trivial facts to generate a complete calculus for first-order logic.

It's a sequent calculus. For details on the web:
`http://wilfridhodges.co.uk/history19a.pdf`.

Typical axioms:

- $(\phi \wedge \psi) \vdash \psi$
- $\forall x \phi \vdash \phi[t/x]$ (t any variable)

Three rules generating new sequents:

- Ibn Sīnā's Principle.
- If $T \vdash \psi$ then $T \cup U \vdash \psi$.
- If $T \vdash \psi$ and for each $\phi \in T$, $U \vdash \phi$, then $U \vdash \psi$.

Ibn Sīnā describes Euclid as a 'syllogistic' mathematician. The implication is that Ibn Sīnā claims he can use syllogisms to justify the logic of all of Euclid's arguments.

His notion of 'justifying' is formally much weaker than ours today (though epistemologically stronger).

His use of instances of Ibn Sīnā's Principle makes this claim definitely more plausible.

(Remark: For him Archimedes was not 'syllogistic'!)

There is an obvious close link between Ibn Sīnā's treatment of assumptions and Frege's treatment of the same topic in his late unpublished 'Logic in mathematics' and his 'Grundlagen der Geometrie' written against Hilbert.

Both Ibn Sīnā and Frege regard assumptions as a way of shortening arguments.
Frege speaks of the 'monstrous length' of propositions if all assumptions were added to them explicitly.

But their aims were different.

Frege wanted each step of a demonstration (*Schluss*, not *Ableitung*) to be a fully meaningful proposition stating the new information gained at that step.

Ibn Sīnā wanted each step to be fully explicit about the logician's intentions, and intuitively convincing.

Someone should certainly compare the approaches of Ibn Sīnā and Frege to making and discharging assumptions.