Could Ibn Sina's logic be undecidable?

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Ibn Sīnā (Avicenna, early 11th century Persian) wrote logic in the Aristotelian tradition.
He introduced several variants that today we would call 'logics'. We can usefully distinguish five of them.
Some are more precisely defined than others.

Logic 1: Aristotelian assertoric logic. This is Aristotle's basic logic of syllogisms, which Ibn Sīnā accepts wholesale and uses a basis for nearly all his other logics. (Rather as modern logicians build their logics on boolean algebra.)
Typical assertoric syllogism:

Some C is a B.
No B is an A.
Therefore not every C is an A.

Ibn Sīnā makes two main adjustments to assertoric logic.

First (excuse technicality!) he introduces an ecthetic proof for *Baroco*. This is important for adaptations to his other logics. It is also a good criterion for identifying Ibn Sīnā's followers in Arabic logic.

Second he introduces syntactic rules for checking the validity of syllogisms (more precisely for checking whether the premises yield a syllogistic conclusion, and if they do, what is the strongest conclusion).

Logic 2: Two-dimensional temporal logic, with a quantification over times. Typical two-dimensional syllogism:

Every C is a B all the time it exists.
Every B is an A all the time it is a B.
Therefore every C is an A all the time it exists.

Logic 3: Alethic modal logic of necessary and possible.
Ibn Sīnā deliberately confuses this with two-dimensional logic, and uses the properties of temporal logic to justify alethic modal arguments.

Logic 4: Propositional logic with quantification over times.
This is an offshoot of two-dimensional logic with wide scope for the time quantifications, and with a free use of negation not found in Western logic before the 19th century.

Logic 5: Logic of finite and infinite sequences.
This logic is problematic. Ibn Sīnā uses it for studying other logics, and also in his proofs of God's existence and uniqueness. Unclear how far it is a well-defined logic at all.

Remark (WH).

It's generally agreed that Ibn Sīnā must have intended his modal logic to be helpful for metaphysical arguments.
So we expect it to be relevant to his most famous metaphysical arguments, such as his proofs of the existence and uniqueness of God. But nobody has identified any place where Ibn Sīnā uses his modal logic in these arguments.

However, he does here use arguments about finite and infinite temporal or causal sequences, that look interestingly like some arguments that he uses in studying the notion of compound proofs. See for example a recent formalisation (for computer checking) of his metaphysical arguments, by Ebnenasir and Tahat at Michigan Technological University (forthcoming).

Fact: In all the cases where Ibn Sīnā's logics have been precisely identified, validity is decidable.
Ibn Sīnā's use of syntactic validity tests for assertoric logic, and his comments on this, show that he regarded decidability of validity for this logic as fundamental for applications of logic in other sciences.

But he seems not to follow any principle that would guarantee decidability.
For example one easily constructs two-dimensional sentences that need at least three distinct variables (so Mortimer's Theorem doesn't guarantee decidability for these).

Maybe some general principle guaranteeing decidability will emerge as we study Ibn Sīnā's logics further.

But it makes sense to look from the opposite end too, and see how much we need to put into Ibn Sīnā's two-dimensional logic to get undecidability.

The answer below uses suggestions from a discussion between Erich Graedel and Wilfrid Hodges in 2013,
in answer to a query from Maarefi.

Ibn Sīnā's two-dimensional logic is in a language $\mathcal{L}$ that is two-sorted with sorts *objects* and *times*; the nonlogical symbols are all binary relations with first sort *objects* and second sort *times*; we say it has sort $(objects, times)$.
There is no equality relation in $\mathcal{L}$.
Ibn Sīnā uses a distinguished relation E:

$Ex\tau$ means 'object x exists at time τ'.

So the logic is monadic in each sort.
Monadic one-sorted logic is decidable (i.e. validity of finite inferences in this logic is decidable).
Does this property extend to the logic $\mathcal{L}$?

Answer: No, at least if we add equality in each sort.

Sketch: Using equality we can write a sentence θ saying that $Ex\tau$ expresses a bijection between the domain of objects and the domain of times.
Then we can encode any binary relation R on times by a binary relation S of sort $(objects, times)$, by:

$$R\alpha\beta \;\Leftrightarrow\; \exists x\,(Ex\alpha \wedge Sx\beta).$$

But by Kalmár 1936, the first-order theory of a single binary relation, even in a language without equality, is undecidable.

We want to move this result closer to the kinds of thing that Ibn Sīnā does actually say in his two-dimensional language.

There is no need for equality on objects.
Probably Ibn Sīnā believed that time is linearly ordered, which we can express with a theory $Th(\leqslant)$ using a relation $\leqslant$ on times:

$$\forall\rho\forall\sigma\forall\tau\;(\rho \leqslant \sigma \wedge \sigma \leqslant \tau \rightarrow \rho \leqslant \tau)$$
$$\forall\rho\forall\sigma\;(\rho \leqslant \sigma \wedge \sigma \leqslant \rho \rightarrow \rho = \sigma)$$
$$\forall\rho\forall\sigma\;(\rho \leqslant \sigma \vee \sigma \leqslant \rho)$$

We write $\eta x\rho$ for the formula

$$(Ex\rho \wedge \forall\sigma(Ex\sigma \to \rho \leqslant \sigma))$$

which expresses that x first comes into existence at the time ρ. We write *Corresp* for the conjunction of the two following sentences, which express a correspondence between objects and times:

$$\forall\rho\exists x\ \eta x\rho$$
$$\forall x\exists\rho\ \eta x\rho$$

Now given a relation symbol Q of sort (*objects*, *times*), we can define a relation R on times by

$$R\alpha\beta \;\Leftrightarrow\; \forall x(\eta x\alpha \to Qx\beta).$$

So every sentence ϕ about R translates into a sentence $\phi^\star$ in terms of η and Q.

Theorem. If ϕ is any first-order sentence about R, then

$$\vdash \phi \quad \Leftrightarrow \quad Th(\leqslant), Corresp \vdash \phi^\star.$$

It follows that two-dimensional logic is not decidable. For if it was decidable, we could determine whether a sentence ϕ about a single binary relation R is logically valid by checking whether the sentence

$$\left(\left(\bigwedge Th(\leqslant) \wedge Corresp\right) \to \phi^\star\right)$$

is logically valid.

The proof of the Theorem is not hard.

Direction $\Leftarrow$: Suppose $\vdash \phi$. By our interpretation of R, ϕ expresses that $\phi^\star$ holds, given any linear relation $\leqslant$ and correspondence η satisfying *Corresp*. So $Th(\leqslant), Corresp \vdash \phi^\star$.

Direction $\Rightarrow$: Suppose there is a model of $\neg\phi$. We check that we can add relations $\leqslant$ and η so as to get a model of $Th(\leqslant)$, *Corresp* and $(\neg\phi)^\star$. But $(\neg\phi)^\star$ is $\neg(\phi^\star)$, so the model shows $Th(\leqslant), Corresp \nvdash \phi^\star$.

Remarks (WH).

1. Very likely the logic $\mathcal{L}$ is undecidable anyway, by an elaboration of Kalmár's proof. This may even be a known fact, but it might be hard to find in the literature, and is likely to be messy to prove.

2. We can get undecidability of validity of sequents even if we limit the sequents to relatively simple sentences, provided we allow arbitrarily many sentences in the sequent. So Ibn Sīnā is not protected against undecidability by the fact that he only uses relatively simple sentences. He himself says you can expect to meet formal arguments with a thousand steps.

Conclusion

Ibn Sīnā's syntactic conditions for validity of assertoric syllogisms, and his comments on these conditions, show that for him one of the most important features of logic is the ability to determine whether a formal inference is valid, i.e. that we have an algorithm for testing validity.

All his well-defined 'logics' have such an algorithm, though he may not have known it in all cases. (He may have had his own private procedures for checking, and perhaps he showed them to some of his students.)

We have seen that the situation may be different for some of his less well-defined logics. If he had understood the possibility of an undecidable logic, would he have taken steps to avoid it? Should he have done, given his declared views about logic?