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Abstract. Ibn Sīnā (11th century, greater Persia) proposed an analysis of arguments by reductio ad absurdum. His analysis contains, perhaps for the first time, a workable method for handling the making and discharging of assumptions in a formal proof. We translate the relevant text of Ibn Sīnā and put his analysis into the context of his general approach to logic.

Ibn Sīnā on reductio ad absurdum

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This paper studies the analysis of reductio ad absurdum by Ibn Sīnā (known to the Latin West as Avicenna), who was born in around 980 in a village near the town of Bukhara on the Silk Road in present-day Uzbekhistan, and died in 1037 after a career spent moving around within the present boundaries of Iran.

References to Ibn Sīnā's writings are to his Arabic texts listed in the bibliography, and are given in the format page.line, where both numbers are arabic; a reference of the form *Qiyās* viii.3 is to Section 3 of book viii of *Qiyās*. References to the text translated from *Qiyās* (Ibn Sīnā 1964) at the end of this paper are marked with an asterisk $\star$.

My special thanks to Amirouche Moktefi with whom I discussed the translation in Section 8 in detail on several occasions; but he is not to be blamed for any errors. Thanks also to Rima Said Al-Balushi, Ahmad Hasnaoui and the referee for comments and information relating either to the translation or to historical issues.

§1. The argument form in question We should start with what Ibn Sīnā calls the 'usual' ($^{c}\bar{a}da$) form of proof by reductio ad absurdum (*Qiyās* (Ibn Sīnā 1964) 410.13*). He gives an example at (53) below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Not not every } C \text{ is a } B}{\text{Every } C \text{ is a } B} \qquad \text{Every } B \text{ is an } A}{\text{Every } C \text{ is an } A} \qquad \text{Not every } C \text{ is an } A}{\perp}}{\text{Not every } C \text{ is a } B} \tag{1}$$

This is only one example. Ibn Sīnā was certainly well aware that reductio arguments can include many more steps than this; see for instance his own example in (28)–(31)

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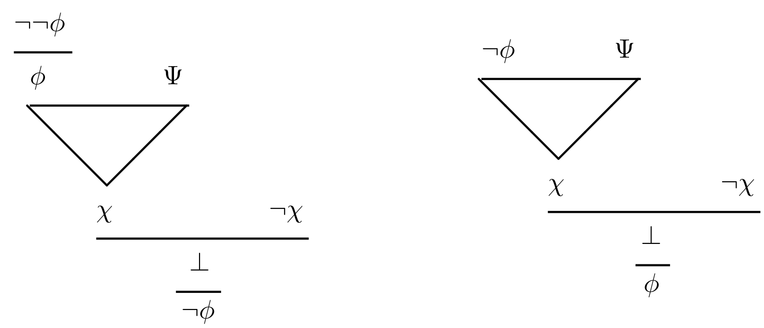
below, and in (27) an example from Euclid that Ibn Sīnā will have been familiar with.. So it seems reasonable to assume that the step from 'Every C is a B' and 'Every B is an A' to 'Every C is an A' is proxy for an arbitrarily complicated derivation, for example a derivation of χ from ϕ and a set of assumptions Ψ. Also Ibn Sīnā gives the form for proving a negated conclusion, and this allows him to remove a double negation at the top of the derivation. If the conclusion was not negated this step would be missing. So we have two general forms:

(2)

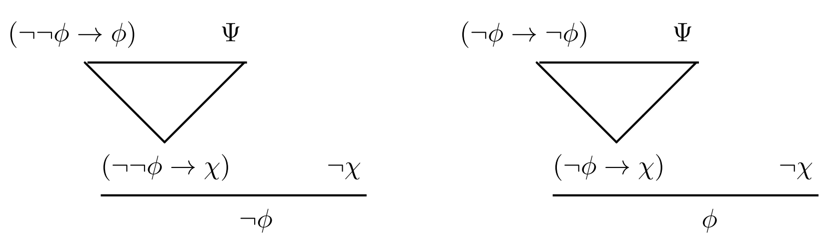


In modern formalisms we would mark that the assumption at top left is discharged at the bottom step. But Ibn Sīnā had no conscious notion of discharging assumptions. I have no idea whether any logician or mathematician of that date or earlier mentions any such notion; certainly I never came across it.

§2. Ibn Sīnā's analysis of the argument Ibn Sīnā believes that when people write the argument (2) above, they normally mean something different. This is an example of a very general claim he makes, that we almost always mean more than we say, and that one task for logicians is to make explicit what we leave unspoken. We come back to this in Section 6 below.

According to Ibn Sīnā, the assumption $\neg\neg\phi$ in the lefthand version of (2) carries through as far as χ, though it is normally not re-stated after it has first been introduced. So when we write χ, we really mean $(\neg\neg\phi \rightarrow \chi)$ (as he says at (54), *Qiyās* (Ibn Sīnā 1964) 410.14f*). Presumably the assumption $\neg\neg\phi \rightarrow$ has to be added at all steps of the derivation that depend on the initial $\neg\neg\phi$. At top left this allows us to replace the derivation of ϕ from $\neg\neg\phi$ by a single formula: $(\neg\neg\phi \rightarrow \phi)$ (as he gives it at (38)). The 'impossible absurdity' represented by $\perp$ now resolves into a derivation of $\neg\phi$ from $(\neg\neg\phi \rightarrow \chi)$ and $\neg\chi$. This and analogous adjustments of the righthand version of the argument transform (2) into:

(3)



For convenience we can call (2) the *surface forms* and (3) the *deep forms.*

The passage from the surface to the deep form has effects on the top, the middle and the bottom of the derivation. At the top, there is no longer any issue of making assumptions; the entire derivation can be read as deductions from given facts. This is pure gain and I say very little more about it (but see Subsections 7.6 and 7.7 below). In the middle, valid inference steps are complicated by having an 'If ...' clause added to the conclusion and one premise. At the bottom, there are no longer any inferences from two contradictory premises χ and $\neg\chi$; instead we have a step of modus tollens.

We should discuss how Ibn Sīnā justifies the middle and bottom parts of the deep form of the derivation. This will need some preliminaries on Ibn Sīnā's proof theory, more precisely on his treatment of assertoric logic. The next section will gather up the main facts that we need.

§3. Ibn Sīnā's assertoric logic You may read that Ibn Sīnā rejected Aristotle's assertoric (non-modal) logic. This is the opposite of the truth. Ibn Sīnā accepted this part of Aristotle's logic lock, stock and barrel, and nearly all the innovations that he made are based on this assertoric logic. In particular Ibn Sīnā's new sentence forms are all got by taking assertoric sentences and applying adjustments, chiefly in the form of 'attachments' that make the sentences more complicated. Also Ibn Sīnā used Aristotle's proof theory for assertoric logic as a template for developing his own proof theory for other kinds of logic. So we need to sketch Aristotle's assertoric logic and some tweaks that Ibn Sīnā made in it.

Aristotle introduced four sentence forms, that we can write as

(4)
Every B is an A.
No B is an A.
Some B is an A.
Not every B is an A.

These are the sentence forms of assertoric logic. Ibn Sīnā tends to refer to them as the 'standard' (*mashhūr*) forms. Aristotle in his *Prior Analytics* i.4–6, (Barnes 1984) pp. 41–47, classified the two-premise valid inference steps (i.e. the 'syllogisms', Arabic *qiyās*) that use the assertoric forms. He presented them as an axiomatic system; I will refer to this system as Aristotle's proof theory.

This proof theory takes some syllogisms as axioms, noting that they are self-evidently valid. Thus one of the axioms is the syllogism

(5) Every C is a B. Every B is an A. Therefore every C is an A.

This syllogism was known to the medieval Latins as *Barbara*; we saw it as a step in (1) above. For those syllogisms that he didn't take as axioms, Aristotle provided proofs by deriving them from axioms. For example in (14) below we will meet the syllogism known to the Latins as *Cesare*:

(6) Every C is a D. No A is a D. Therefore no C is an A.

Aristotle proved this as follows. Assume the two premises. Then by the second premise, no D is an A (this step is called 'conversion', Arabic *aks*). But an axiom tells us that if every C is a D and no D is an A, then no C is an A. (*Prior Analytics* i.5, 27a5–7, (Barnes 1984) p. 43)

Aristotle also remarked (*Prior Analytics* i.5, 27a15, (Barnes 1984) p. 43) that *Cesare* can be proved by reductio ad absurdum. He presumably meant the following argument or some slight variant of it. Assume the two premises and suppose for contradiction that some C is an A. An axiom tells us that if some C is an A and no A is a D, then some C is not a D. But this contradicts the premise that every C is a D.

Six reports of assertoric logic and its proof theory appear in Ibn Sīnā's surviving logical works, and four of them follow Aristotle down to the fine details of the proofs. (A fifth uses exactly the same formalism to develop a system of propositional logic; the sixth presents the assertoric logic but is a little abbreviated.)

Ibn Sīnā does make one systematic addition to Aristotle's assertoric proof theory. Among the syllogisms that Aristotle didn't take as axioms, there is just one where Aristotle's only method for proving it was to use reductio ad absurdum. This was the syllogism *Baroco*:

(7) Not every C is a B. Every A is a B. Therefore not every C is an A.

Aristotle argued: Assume for contradiction that every C is an A. Then by applying *Barbara* to this supposition and the second premise, we deduce that every C is a B, and this contradicts the first premise. (*Prior Analytics* i.5, 27a36–27b1, (Barnes 1984) p. 44).

Ibn Sīnā mentions this proof by reductio ad absurdum, but in all six of his reports he also gives a new proof which doesn't involve reductio. He argues as follows (for example at *Qiyās* (Ibn Sīnā 1964) 116.10—I have slightly expanded some details). Suppose not every C is a B. Write D for the property 'a C but not a B'. Then some C is a D, and no B is a D. The second premise states that every A is a B, and this with 'No B is a D' tells us (by an axiom) that no A is a D, and so (by conversion) no D is an A. Then another axiom tells us that if some C is a D and no D is an A, then some C is not an A, in other words, not every C is an A, as required. This is a type of proof which Ibn Sīnā calls *iftirāḍ*, translating Aristotle's 'ecthesis'. He also calls it *ta*^{c}*yīn*, 'specification', because it involves introducing a new letter with a meaning specified in terms of the existing letters.

This proof of *Baroco* by ecthesis seems to be Ibn Sīnā's own innovation. From the start it became a trademark of the Ibn Sīnā tradition in logic: we find it in Bahmanyār, Sāwī, Abū al-Barakāt and Rāzī, but not in Ibn Sīnā's opponent Ibn Rushd. It plays a role in Ibn Sīnā's adaptation of Aristotle's arguments to temporal and modal logic. But already in the proof theory of assertoric logic it has the effect that Ibn Sīnā never needs to use reductio ad absurdum. We will come back to this point.

A second new feature of Ibn Sīnā's assertoric logic is not so much a change as a comment. Ibn Sīnā believed that human minds have a piece of dedicated machinery, which he calls the *bāl*, for processing strings of symbols so as to perform deductions. For example one activity of the *bāl* which he mentions in a number of places is the following. Two sentences are fed into the *bāl* one after the other, and the *bāl* scans them to see whether there are any common elements between the two sentences. The *bāl* does this job best, he claims, if the common element appears near the end of the first sentence and near the beginning of the second. At *Qiyās* (Ibn Sīnā 1964) 410.2* below he suggests a name for this activity of finding common elements:

'idġām. The name comes from linguistics and refers to the way in which similar sounds at the end of one word and the beginning of the next can be combined. (I translated it 'unification', taking a term from modern logic. You can find out more about *'idġām* by going to the internet and looking up the procedures of Qur'anic recitation. There is further information about the *bāl* in (Hodges 2011).)

The *bāl* is equipped to perform all the syllogisms of assertoric logic. In each case it carries out a unification, removes the unified parts and recombines the remaining pieces of the sentences into a new sentence, which it outputs as conclusion. Ibn Sīnā describes inferences that work this way as 'recombinant' (*iqtirānī*). For example in *Barbara*, (5) above, the inputs are 'Every C is a B' and 'Every B is an A'. Unification finds B in both sentences. The *bāl* removes both occurrences of B and reassembles the remaining letters to form 'Every C is an A'.

Ibn Sīnā points out that the sequence of operations used to perform assertoric inferences—unifying, removing, recombining—works also for some propositional inferences. An example is

(8) Whenever p, then q. Whenever q, then r. Therefore whenever p, then r.

So this is an example of an *iqtirānī* syllogism that is not assertoric. If Ibn Sīnā had been Wallis, or more famously Boole, he would also have pointed out that this inference can be reduced to assertoric *Barbara* by a paraphrase:

(9)
Every time t such that $p(t)$ is a time such that $q(t)$.
Every time t such that $q(t)$ is a time such that $r(t)$.
Therefore every time t such that $p(t)$ is a time such that $r(t)$.

It is possible that he simply didn't notice this reduction. But when justifying inferences, he prefers to give justifications that stay close to the inference and don't invoke other kinds of reasoning. We come back to this point in Section 6 below.

§4. Sequent rules and the middle of the derivation In the centre of the derivations (2) there are one or more steps

(10) $$\phi, \psi \vdash \chi$$

that are replaced in the deep derivations (3) by steps

(11) $$(\theta \to \phi), \psi \vdash (\theta \to \chi).$$

We ask: is Ibn Sīnā right to believe that if (10) was valid, then (11) is also valid? And if he is right, then how does he know this logical fact, and how does he justify it to himself and his readers?

In *Qiyās* viii.3, translated in Section 8 below, Ibn Sīnā gives no hint of the answers to these questions. But in fact he has already devoted a whole section of *Qiyās*, Section vi.4, to what is essentially the same logical problem, working in detail through over fifty examples. (*Qiyās* vi.4 is translated in (Shehaby 1973). Shehaby translates Ibn Sīnā's sentence forms using translations that were proposed by Nicholas Rescher before *Qiyās* had become available, and now that we have *Qiyās* I think we can see that Rescher's translations don't all work. I hope to put this in print soon, but meanwhile the reader might have trouble verifying some of the statements below from Shehaby's translation.)

It has to be said straight away that Ibn Sīnā has certainly got himself out of his depth here. Correct answers to the questions that he poses in *Qiyās* vi.4 would need an understanding of both variables and metavariables that he didn't have. But he was moving into new territory; to the best of my knowledge he was the first logician to have made a systematic study of rules like the one taking the sequent (10) to the sequent (11)—let us call them 'sequent rules'. Moreover the examples that he studies in *Qiyās* vi.4 are a level more complicated than the propositional rule taking (10) to (11), because they involve another of his innovations, namely quantification over time.

But by good fortune Ibn Sīnā is right. The sequent rule taking (10) to (11) is perfectly correct, and so are all the applications of similar sequent rules in *Qiyās* vi.4—setting aside a very few problems about the correct reading of the manuscript text. Furthermore we will see below that we can use modern logic to formulate a reasonably straightforward sequent rule that we can demonstrate is correct and covers all the cases considered by Ibn Sīnā. So he had thoroughly sound intuitions, even if he lacked the logical machinery to give cogent proofs of them.

Consider a typical example (*Qiyās* (Ibn Sīnā 1964) 327.13–17).

(12) Whenever r, then every C is a D;
and no A is a D.
So whenever r, then no C is an A.

It can be demonstrated by converting the predicative premise. It can also be demonstrated as follows:

(13) Whenever r, then C is a D;
and no A is a D.
But whenever C is a D and no A is a D, then no C is an A.
It yields: Whenever r, then no C is an A.

(Here r abbreviates Ibn Sīnā's 'H is Z'. This abbreviation makes no difference to the arguments below.) Ibn Sīnā has taken an assertoric syllogism, in this case *Cesare*:

(14) Every C is a D. No A is a D. Therefore no C is an A.

He has added 'Whenever r then' to the beginning of the first premise and the conclusion. All the examples in *Qiyās* vi.4 follow this same general pattern, starting with an assertoric syllogism and attaching pieces of various kinds. One of the premises stays unchanged, and he calls this premise the 'predicative premise'. We will refer to the starting syllogism as the 'underlying assertoric syllogism'.

How does Ibn Sīnā justify (12)? In this passage he offers two justifications. The first is 'by converting the predicative premise'. This means: use the fact that 'No A is a D' says the same as 'No D is an A'. That changes the underlying assertoric syllogism to *Celarent*; Ibn Sīnā has already considered this case at *Qiyās* (Ibn Sīnā 1964) 326.12, where he considers it too trivial to deserve a detailed explanation.

The second justification is more useful to us. This appears as the argument (13). The argument consists of (12) but with a third premise added. This third premises is simply the syllogism *Cesare* written as an implicative formula. So essentially all that Ibn Sīnā has done in (13) is to remind us of the underlying assertoric syllogism.

What does he expect us to do with (13)? The conclusion doesn't follow from the three stated premises by any obvious sequence of rules that Ibn Sīnā has already described. Does he expect us to invent the rules? It would probably be a different sequence of rules for each underlying assertoric syllogism. So it seems very unlikely that he expects this.

A more plausible answer is that he wants us to check that we believe *Cesare*, and then in the light of that, check that we find it convincing to add the prefix 'Whenever r' to one premise and the conclusion of *Cesare*. In other words, this is an appeal to intuition, strengthened by the fact that he expects us to check the same intuition in a lot of different cases.

This reading of his argument is reinforced by the fact that it is an example of a pattern that keeps appearing in the more advanced parts of Ibn Sīnā's logic. Many of his sentences consist of underlying simple sentences with added 'attachments' or 'conditions', for example the modalities 'Necessarily' or 'Possibly'. When verifying arguments that use these more complicated sentences, Ibn Sīnā normally (I think in fact always) restricts himself to arguments that would be valid if the attachments were removed, before checking that the arguments are still valid with the attachments included. For example the only modal syllogisms that he checks are those whose underlying assertoric syllogisms are already known to be valid. There are metatheorems that make this a sound policy in most cases. We don't know how far Ibn Sīnā was aware of these metatheorems, and how far he was simply following Aristotelian practice.

But Ibn Sīnā certainly was aware of this division of labour, separating off the checking of the underlying syllogisms from what he calls 'taking care of the conditions'. He refers to it at *Qiyās* (Ibn Sīnā 1964) 226.11, 325.7–10 and 472.9–11 for example. He also refers to it in a key passage of his *Autobiography*, where he describes his own logical procedures. It's widely recognised that Ibn Sīnā is setting himself up as a paradigm here, not just describing facts about his past life. So the details matter.

(15) I put together in front of me [sheaves of] scratch paper, and for each argument that I examined, I recorded the syllogistic premisses it contained, the way in which they were composed, and the conclusions which they might yield, and I would also take into account the conditions of its premisses until I had Ascertained that particular problem. (*Autobiography* (Gutas 2014) pp. 16f.)

Note the phrase 'take into account the conditions'. I omitted Gutas' explanatory phrase '[i.e. their modalities]'; Gutas is certainly right that modalities count as conditions here, so that Ibn Sīnā's statement is meant to include modal reasoning. But there is no reason to exclude other kinds of condition, including the conditional clauses of Ibn Sīnā's account of reductio ad absurdum.

So I will assume that Ibn Sīnā thinks that he and his readers can intuit that if (14) is valid, then so is (12). Unsurprisingly, the things that Ibn Sīnā proves by intuition are often much less rigorously stated than the things that he proves formally. In the present case he glosses over the implications of 'Whenever'. For the first premise and the conclusion of (12) to make sense, the sentences 'Every C is a D' and 'No C is an A' must have some dependence on time; for example D could be read as 'D at time t', and A likewise. But then we have to allow the second premise to contain a reference to time too, and in general the validity of (12) will require that the second premise is understood as being always true. So strictly (14) needs to be read as

(16) At time t, every C is a D. Always, no A is a D. Therefore at time t, no C is an A.

This is not straight assertoric *Cesare*, though it is valid and we could allow it as a temporal variant of *Cesare*.

Further discussion of this example takes us beyond the scope of Ibn Sīnā's logical tools, so we may as well go straight to a modern formulation. Consider operations $\delta(-)$ that take formulas to formulas. For which such operations δ do we have the following sequent rule?

$$\eta, \Psi \vdash \theta \quad \Rightarrow \quad \delta(\eta), \Psi \vdash \delta(\theta) \tag{17}$$

In *Qiyās* (Ibn Sīnā 1964) vi.4 Ibn Sīnā shows that this sequent rule holds if the inference $\eta, \Phi \vdash \theta$ is any of the Aristotelian assertoric syllogisms (with an adjustment to include t, as discussed above), and δ is either of the following two operations:

$$\begin{array}{l} \delta(\psi) = \forall t\ (\phi \to \psi). \\ \delta(\psi) = \exists t\ (\phi \wedge \psi). \end{array} \tag{18}$$

The derivations in (3) apply the sequent rule but with

$$\delta(\psi) = (\phi \to \psi). \tag{19}$$

This differs from the first δ of (18) because it lacks the universal quantification over times; in other words it expresses 'If' rather than 'Whenever'. We will come back to this difference in the next section.

I leave it to the reader to check that the following principle holds for first-order sequents in any standard calculus:

(20) **'Ibn Sīnā's Principle'**: Suppose T is a set of formulas and η, θ are formulas. Let $\delta(p)$ be a first-order formula containing a propositional variable p which occurs only positively in $\delta(p)$ and doesn't occur in the scope of any quantifier on a variable free in some formula of T. If $T, \eta \vdash \theta$ then $T, \delta(\eta) \vdash \delta(\theta)$.

This principle justifies the sequent rule (17) with any of the δ of (18) and (19). But of course Ibn Sīnā couldn't have stated the principle as we stated it in (20)—not least because he had no workable notion of scope ((Hodges 2015) Sections 6–9).

§5. Propositional logic and the bottom step The modus tollens argument at the bottom of the derivation (3) creates no problems. By Ibn Sīnā's time this was a well-accepted form of argument; Ibn Sīnā discusses it in detail at *Qiyās* (Ibn Sīnā 1964) 395.8, under the name of the 'fourth standard duplicative mood'.

It's just as well for Ibn Sīnā that his deep form (3) removes the inference step

$$\chi, \neg\chi \vdash \bot \tag{21}$$

which appeared at the bottom of the surface form. At *Qiyās* (Ibn Sīnā 1964) 547.13f he claims that the *bāl* is incapable of accepting inputs of the form $\chi, \neg\chi$. We needn't read him as saying that (21) is an invalid inference. More likely his point is that there doesn't seem to be any way of reading (21) as a real-life inference. Faced with premises χ and $\neg\chi$, our first reaction is to detach ourselves from the premises and observe that the two propositions are incompatible.

Unfortunately there is a more serious issue of propositional logic connected with (19), the operator δ that expresses 'If'. To explain it we must sketch the historical background of Ibn Sīnā's propositional logic. I take the opportunity to explain some of his terminology along the way.

Ibn Sīnā's propositional logic has several layers. Probably he formulated them at different stages of his logical development, though in *Qiyās* he presents them all simultaneously. The most primitive layer, called PL1 in (Hasnawi & Hodges 201-) Section 4.3, consists mostly of material that Ibn Sīnā inherited from his predecessors. It revolves around the two sentence forms

(22) If ϕ then ψ.
Either ϕ or ψ.

where ϕ and ψ are sentences. The first of these two forms was called in Arabic *muttaṣil*, to indicate that the sentence expresses a 'connection' or 'meeting' (*waṣl*) between ϕ and ψ. The second was called *munfaṣil*, to indicate that it expresses a 'difference' (*faṣl*) between ϕ and ψ, no doubt because the disjunction tended to be taken as exclusive. The two sentence forms together were called 'conditional' (*sharṭī*); it was held that the first form makes ϕ a sufficient condition for ψ to hold, and the second form makes the falsehood of ϕ a necessary condition for ψ to hold.

Ibn Sīnā gives no semantics for the two forms. Instead he classifies their interpretations in terms of the inferences that they enter into. For example the *muttaṣil* form always allows modus ponens: 'If ϕ then ψ. But ϕ. Therefore ψ.' Sometimes the implication is taken to run both ways, so that from 'If ϕ then ψ' and ψ we can deduce ϕ. These are both examples of inference patterns where the two premises are a conditional sentence and one of its clauses (possibly negated as in modus tollens), and the conclusion is the other clause (again possibly negated). Ibn Sīnā calls inference steps of this kind *duplicative* (*istiṯnā'ī*). This name is older than Ibn Sīnā; its origin seems to have been a supposed analogy with an Arabic grammatical construction called 'exception', *istiṯnā'*. But the inference steps in question have nothing to do with exceptions. Perhaps Ibn Sīnā just kept the name that he learned from earlier logicians. But we can give the word more sense if we suppose that Ibn Sīnā went back to its etymology and read it as 'repetition' or 'duplication'; then it could be used to express that the second premise *repeats* or *duplicates* a clause of the first premise. I hope this is what he meant, but we may never know.

Ibn Sīnā's temporal logic inspired him to recast propositional logic, using quantification over times in analogy with the quantifiers in assertoric sentences. He did this first with the *muttaṣil* sentences, producing the layer called PL2 in (Hasnawi & Hodges 201-) Section 4.3. Instead of the one form 'If ϕ then ψ' and some variants of it, he now had four *muttaṣil* forms

(23) Whenever ϕ then ψ.
Whenever ϕ then not ψ.
There is a time at which both ϕ and ψ.
There is a time at which ϕ but not ψ.

He presented this logic of temporal *muttaṣil* sentences in *Qiyās* (Ibn Sīnā 1964) vi.1, in a form that makes it clear that it is isomorphic to assertoric logic, even down to its proof theory in Aristotle's style (and of course with the ecthetic proof of *Baroco*). This was an elegant development, but it caused problems for understanding reductio ad absurdum.

The first problem is as we noted earlier, that the quantification of times gives 'Whenever ϕ then ψ' rather than 'If ϕ then ψ'. Ibn Sīnā glosses over this problem. No doubt he regards 'If' and 'Whenever' as closely similar in their logical behaviour. But there is a more serious problem, as follows.

Ibn Sīnā understood the assertoric form 'Every B is an A' as implying that there is at least one B (Hodges 2012). In fact this implication is needed for making some of Aristotle's proof theory work. So Ibn Sīnā had to carry the corresponding implication over to *muttaṣil* sentences in PL2. In other words, the sentence 'Whenever ϕ then ψ' had to be read as implying that there is a time when ϕ is true. This assumption wreaks havoc with applications of reductio ad absurdum. We want to be able to prove necessary truths χ by arguing that if not-χ then some contradiction follows; but if χ is a necessary truth then there is no time when not-χ. Ibn Sīnā does show some awareness of this issue, though not in the passage of *Qiyās* that we are dealing with, and what he says about it is obscure.

In the next layer, PL3, Ibn Sīnā sets out to give a temporal content to *munfaṣil* sentences as well. For reasons too complicated to go into here, this new context makes it unreasonable to keep assuming that 'Whenever ϕ then ψ' implies that ϕ is true at some time. So Ibn Sīnā drops the implication, and this leaves him free to present his analysis of reductio ad absurdum without having to refer to any such implication. But it's undeniable that these various changes of direction lead to a messy presentation in *Qiyās*.

The name 'conditional' is hardly appropriate for the third and fourth sentences in (23) above, and the misfit increases if we take on board Ibn Sīnā's quantified *munfaṣil* sentences as well. Ibn Sīnā himself was well aware of this misfit. In his *Easterners* (Ibn Sīnā 1910) 61.7–12 he suggested that *sharṭī* should be understood as meaning, not that there is a condition involved, but that there are subclauses which are not asserted when the sentence as a whole is asserted. In line with his suggestion I translate *sharṭī* as 'propositional compound'. Propositional compound sentences are contrasted with the 'predicative' (*ḥamlī*) sentences that contain only one clause. Assertoric sentences are predicative, but there are predicative sentences that are not assertoric, for example assertoric sentences with a modality added.

There are probably no ideal translations of *muttaṣil* and *munfaṣil*. For the former, suggested translations include 'conjunctive', which is too specific about the logical operator involved, and 'connective' which has no logical content at all. I translate *muttaṣil* as 'meet-like' and *munfaṣil* as 'difference-like'; these translations are almost literal, and they suggest connections to boolean meet and difference without being too precise.

§6. Putting the pieces together For Ibn Sīnā, one of the main tasks of a logician is to take an argument expressed in natural language, and reduce it to a succession of steps that are instances of valid formal inference steps. He calls this task 'analysis' (*taḥlīl*—see (Hodges 2010) p. 385f for a translation of one of Ibn

Sīnā's accounts of *taḥlīl*, and (Hasnawi & Hodges 201-) Section 3.1 for the notion of *taḥlīl* in Arabic logic generally). The case that concerns us here is where the argument being analysed is by reductio ad absurdum.

Analysis often involves paraphrasing the original argument. Ibn Sīnā makes it a general rule that any paraphrasing should be as light as possible. Any paraphrase creates a distance between the original argument and the formal argument used to justify it, and we need to be sure that no error or logical gap has crept in during the paraphrase. Ibn Sīnā doesn't say so explicitly, but clearly in the present case we need to be sure that the paraphrase doesn't itself rest on an undeclared use of reductio ad absurdum, or the analysis would be circular.

A point that Ibn Sīnā does make, frequently, is that the paraphrase may need to make explicit some things that were taken for granted in the original argument. He particularly emphasises that the author of the original argument may have intended some unspoken 'conditions' or 'attachments' to the premises and the conclusion. A typical example is at *Išārāt* (Ibn Sīnā 2000) i.3.10, 83.15 (Inati p. 89): '... if it is said that C is a father, you should take care of the question "[father] of whom?" '.

Ibn Sīnā's analysis of the reductio argument falls neatly into this framework. He finds a 'condition' that was intended but not explicitly stated, and in the deep form he makes it explicit. Moreover this is the only change that he makes to the original argument. He himself stresses this fact in the passage translated in Section 8 below. Thus:

(24) ... there is no need for any contrived and elaborate explanation, in order to give an analysis of the complete form of the syllogism of absurdity, and [to determine] how many syllogisms are needed to complete it. There is no need for the kind of lengthy exposition of [this syllogism] that one finds in the literature. (*Qiyās* (Ibn Sīnā 1964) 408.9–11⋆)

(25) All of these kinds of mutilation, and [these] syllogisms that are hidden and not explicit, lengthen the discussion but give us no new information. [By contrast] the account we have given is exactly the absurdity syllogism itself, no more and no less. (*Qiyās* (Ibn Sīnā 1964) 410.8.10⋆)

The claim that Ibn Sīnā gives 'exactly the absurdity syllogism itself, no more and no less' is exaggerated, but we can see a grain of truth. The only changes that he makes are to bring to the surface a feature that was always intended. He adds no notions that are not in the intended argument itself. In particular there is no chance that any unacknowledged use of reductio has slipped in during the paraphrase.

In case you are thinking that nobody would want to offer a more complicated analysis of reductio anyway, compare this from an explanation of reductio by Augustus De Morgan:

(26) The form in which Euclid argues, supposes an opponent; and the whole argument then stands as follows. "When X is Y, you grant that P is Q; but you grant that P is not Q. I say that X is not Y. If you deny this you must affirm that X *is* Y, of which you admit it to be a consequence that P *is* Q. But you grant that P is not Q; therefore" (etc. etc.) ((De Morgan 1836) p. 5)

This seems a classic example of a 'contrived and elaborate explanation'.

We need some justification for Ibn Sīnā's claim that his added condition was intended but not made explicit. A natural place to look is Euclid's *Elements*, which Ibn Sīnā regarded as a paradigm example of syllogistic reasoning (e.g. *Qiyās* (Ibn Sīnā 1964) 433.6–8.) Here for instance is a literal rendering of a standard medieval Arabic text of Proposition 27 of Euclid *Elements I*:

(27)

> When a straight line lies across two straight lines so that the two symmetrically-opposite angles are equal then the two lines are parallel.
> ...
> Demonstration: If the two are not parallel then when they are both extended on one of the two sides, they meet. So we extend them on the side BD so they meet in a point K if that is possible, so the angle AHT external to the triangle KTH is greater than the internal angle KTH, as was proved in the demonstration of 16 of i, and this is absurd. (*Codex Leidensis* (Besthorn & Heiberg 1893) 114–116)

The reductio assumption is introduced at the beginning of the Demonstration with an 'If' (*'in*) rather than a 'Suppose' (*li-yakun*). As Ibn Sīnā says, this assumption is not repeated anywhere, but we can see that it remains in force through a succession of steps and a change of sentence. Similar examples are easy to find.

When the logician has analysed the argument down to a succession of steps that can be directly justified by formal inferences (syllogisms, conversions etc.), it follows that the argument is valid. A person who accepts the premises can be persuaded to accept the conclusion by going step by step through the formal inferences; if these are not obviously valid, the reasoner can move sideways into the proof theory and derive them from steps that are obviously valid. Normally this would be an end of the matter. But in the present case we also need to check that the formal derivations in the proof theory don't rely on reductio ad absurdum, or at least that any such uses of reductio can be analysed away without circularity.

Here we find ourselves in a position that is rather characteristic of Ibn Sīnā. He doesn't himself mention the need to avoid circularity. But then when we look closer, we find that he has set things up as required. He seems to have anticipated the question, though there is always a possibility that it was a lucky accident.

In the case of the derivations in (3), we saw in the previous section that Ibn Sīnā tends to present his justifications in two steps. The first step is to confirm that the underlying assertoric inference is valid, and the second step is to check that adding the condition does nothing to damage the assertoric inference. We saw in Section 3 that the first step will never need reductio ad absurdum, because Ibn Sīnā himself has introduced a change into Aristotle's schedule that allows all assertoric inferences to be justified without using reductio. The second step is a direct intuition, so again reductio is not needed.

You might think that there was no need for Ibn Sīnā to invoke reductio anyway to justify the use of his sequent rule for δ as in (11). But in fact, when Ibn Sīnā moves in *Qiyās* vi.5 to consider another group of sequent rules, he uses reductio ad absurdum freely throughout the section. He gets a benefit from doing this, namely that he no longer has to appeal to direct intuition; the arguments that he gives in *Qiyās* vi.5 are completely formalised. Here is one example. He is justifying the syllogism

(28) [Always] every C is a B.
There is a time when it is true both that every B is an A, and that r.
Therefore there is a time when it is true both that every C is an A, and that r.

The relevant $\delta(\psi)$ here has the form $\exists t(\psi \wedge \phi)$, and the sequent rule is applied to a variant of *Barbara*. He argues by reductio ad absurdum. Suppose the conclusion fails; then

(29) It is always true that if every C is an A then not r.

From (29) and the second premise it follows that

(30) There is a time when it is true both that every B is an A and that not every C is an A.

Switching the two clauses and applying *Baroco* inside the time quantifier,

(31) There is a time when not every C is an A.

But this contradicts the first premise. (*Qiyās* (Ibn Sīnā 1964) vi.5, 339.5–7.)

In short, Ibn Sīnā uses reductio ad absurdum freely when he can. But he refrains from using it in *Qiyās* vi.4, precisely so that the arguments of *Qiyās* vi.4 are available to justify reductio ad absurdum without circularity.

The arguments of *Qiyās* vi.4 serve to support Ibn Sīnā's analysis of reductio arguments only when the underlying argument is an assertoric syllogism. If the underlying argument is of a different kind altogether, then one might argue that Ibn Sīnā still needs to justify the appropriate cases of 'Ibn Sīnā's Principle'. Since Ibn Sīnā is calling on intuitions rather than formal derivations to justify his sequent rule, he could reply that the variety of assertoric arguments is already a strong enough basis for accepting the passage from (10) to (11) as an intuitively sound universal rule.

§7. Loose ends It seems that everything in Ibn Sīnā has multiple connections. The following topics deserve to be followed up further, but there was no space in this paper.

7.1. Comparison with Frege Ibn Sīnā's analysis has a remarkable amount in common with Frege's analysis of reductio ad absurdum in his 'Logik in der Mathematik' ((Frege 1969) p. 265), and with Frege's treatment of assumptions and their discharging in his 'Grundlagen der Geometrie' ((Frege 1906) pp. 379–381). Ibn Sīnā's insistence at (24) and (25) that his analysis stays close to the original argument could also be set alongside passages where Frege points to the danger of missing out essential steps in an argument. But the aims of the two logicians were certainly not all the same.

7.2. Who is under attack? Ibn Sīnā mentions three other approaches to analysing reductio arguments, and rejects them. The third (in [8.3.6]) closely matches Ibn Sīnā's own summary at *Qiyās* (Ibn Sīnā 1964) 192.9–11 of Aristotle's use of reductio ad absurdum in *Prior Analytics* i.15, 34a34–34b2, (Barnes 1984) p. 55 (except that Ibn Sīnā swaps the major and minor premises, which is irrelevant to the point at issue). The summary is not very close to Aristotle's own text in either Greek or Arabic, but its question about which premise caused the absurdity is a good match for Philoponus' commentary (Philoponus 1905) 172.3–13 on the passage. Nevertheless Ibn Sīnā does say at *Qiyās* 192.3 and 192.12 that he takes the argument from Aristotle. This has implications for the interpretation of the passage at *Qiyās* p. 192 and some of the surrounding arguments. Some medieval and modern commentators have assumed that Ibn Sīnā was expressing his own opinion in these arguments; but his remarks in [8.3.6] make clear that he dissociates himself from the version that he quotes there.

It could also be useful to know who were the sources of the other two approaches that Ibn Sīnā rejects.

7.3. Ibn Sīnā's empiricism There is a growing literature on Ibn Sīnā's 'empiricism' as an aspect of his theory of knowledge; see for example (Gutas 2012). To the best of my knowledge, this literature has not yet touched on Ibn Sīnā's procedures for discovering logical facts, but it should do. The proof that he gives for cases of 'Ibn Sīnā's Principle' is a prime example. We have mentioned two aspects of this proof. The first is that Ibn Sīnā proves the general rule, not by a general argument, but by examining many special cases. Alfred Tarski was still using essentially the same approach in 1931 for a specific purpose which he describes as follows:

(32) Si nous désirons acquérir la certitude subjective de la justesse matérielle de la déf. 10 et de sa conformité à l'intuition, sans sortir du domaine des considérations strictement mathématiques, nous sommes contraints de recurrir à la voie empirique. ((Tarski 1931) p. 229)

Briefly, Tarski wants to describe how we can come to accept a metamathematical principle with 'subjective certainty', using only methods from mathematics. Ibn Sīnā in his case wants to bring us to subjective certainty in relation to a principle about inferences, using only the inferences themselves. In practice Tarski and Ibn Sīnā do pretty much the same thing: they both invite us to sample a number of instances and check that we are convinced that the higher-level principle is true of each instance that we sample. Tarski distinguishes this approach as 'empirical'. Ibn Sīnā would probably describe it as 'experience' (*tajriba*) or 'examination' (*imtiḥān*), two notions that appear regularly in his theory of science. (See (Hodges 2008) pp. 98, 111f for more comments on Tarski's 'voie empirique'.)

In each individual instance, Ibn Sīnā verifies the rule not by calculation—as Tarski does in his case—but essentially by staring hard at it. Appeals to intuitive certainty play a much larger role in Ibn Sīnā's logic than they do for us today. He reinforces these appeals with requirements of professionalism: we should check many examples for ourselves, and check them right down to bottom level.

7.4. Applying 'IS's Principle' down a branch To apply the Principle down a branch of a derivation, with the same δ at each step, it suffices to apply it separately at each step, because the new conclusion of one step is exactly the new premise of the next step:

$$(33)\qquad \cfrac{\cfrac{\begin{matrix}\vdots\\ \phi_1\end{matrix}\qquad \psi_1}{\phi_2}\qquad \psi_2}{\begin{matrix}\phi_3\\ \vdots\end{matrix}} \qquad\Rightarrow\qquad \cfrac{\cfrac{\begin{matrix}\vdots\\ (\theta\to\phi_1)\end{matrix}\qquad \psi_1}{(\theta\to\phi_2)}\qquad \psi_2}{\begin{matrix}(\theta\to\phi_3)\\ \vdots\end{matrix}}$$

We took this fact for granted in drawing the derivations (3). But it is not a triviality. When Arnauld and Nicole in the *Port-Royal Logic*, (Arnauld & Nicole 1664) iii.13, pp. 274–9 gave a rule for discharging assumptions, possibly for the first time in European logic, they gave it in a form that applies only to a one-step argument. It doesn't propagate down a branch in the same way as Ibn Sīnā's version. Small points like this could be evidence that Ibn Sīnā did write out complex derivations on his dustboard or scratch paper.

7.5. Iterating 'IS's Principle' By iterating 'Ibn Sīnā's Principle' we can apply logical manipulations at any syntactic depth within a sentence—notoriously this was impossible in Aristotelian calculi (Hodges 2009). In early 14th century Paris, Walter Burley ((Burley 1955) p. 68 2nd para.), gave some examples that tend in the same direction as 'Ibn Sīnā's Principle'. But neither Ibn Sīnā nor Burley show any inclination to iterate their methods. The earliest example that I know of a logical truth or inference whose proof needs serious digging down into the syntactic depths is a sentence given by the linguist Al-Sakkākī some two hundred years later than Ibn Sīnā ((Sakkākī 1987) 493.15–18). It boils down to the following propositional tautology:

$$(34)\qquad ((p \leftrightarrow q) \to (((p \to q) \to (q \to p)) \to ((\neg p \to \neg q) \to (\neg q \to \neg p))))$$

A proof of this formula is within easy reach of the sequent rule taking (10) to (11). Turning to more recent logic, it may be significant that in *Grundgesetze I* (Frege 1893) Section 14 Frege observes that (in our notation) modus ponens $(\psi \to \chi), \psi \vdash \chi$ generalises to $\delta(\psi \to \chi), \psi \vdash \delta(\chi)$ where

$$(35)\qquad \delta(\psi) = (\phi_1 \to (\phi_2 \to \ldots \to (\phi_n \to \psi)\ldots)$$

for any n.

7.6. Trivial deductions There is a problem with the formula at top left of the derivations in (3): $(\neg\neg\phi \to \phi)$ or $(\neg\phi \to \neg\phi)$. We know that Ibn Sīnā regarded the first formula as expressing a true principle, and he could hardly have said less for the second one. But for Aristotelians there was an issue, that Aristotle had said that the conclusion of a syllogism should be something 'different from' the premises. If $(\neg\phi \to \neg\phi)$ is going to be used so as to replace $\neg\phi$ in some other formula by $\neg\phi$, then nothing 'different' will have been proved. Ibn Sīnā discusses the question at *Qiyās* (Ibn Sīnā 1964) 69.13, in connection with what he calls an 'ugly example' (*Qiyās* 66.14f):

(36) If there is movement then there is movement;
there is movement.
Hence there is movement.

His preferred response is that if the conclusion of a syllogism has to be 'something different', then this example is not a syllogism. But he is perfectly happy to say that there are valid arguments that are not syllogisms; for example all valid one-premise arguments.

7.7. Double negation The double negation at top left of the first derivation in (2) deserves a remark. Probably Ibn Sīnā wrote it that way because he reckoned that a reductio argument normally begins by assuming that something is false. But in any case he had no problems with deducing ϕ from $\neg\neg\phi$. Not all logicians of his date had a sentence negation operator. But Arabic has one built in: *laysa*, which negates sentences when it is applied at the beginning of them.

On page 308 of the otherwise very excellent *Oxford Handbook of Medieval Philosophy* (Marenbon 2012) Chris Martin remarks that Ibn Sīnā 'seems not to ... note the possibility that [sentence negation] might be iterated'. One has to be a little careful here: a *laysa* at the beginning of a sentence could be a sentence-internal verb that by the rules of Arabic happens to be written at the beginning. But when an initial *laysa* is followed by *kull* 'every', then it has to be a sentence negation, because in this position it always includes *kull* within its scope. Now the manuscript text at *Qiyās* (Ibn Sīnā 1964) 410.13* needs correction, but almost certainly it should read *laysa laysa kull*. So here we have an iterated sentence negation, and Ibn Sīnā is saying precisely that this double *laysa* can be dropped. There are also examples of *laysa laysa kull* in *Išārāt* (Ibn Sīnā 2000) i.5.4 end and i.8.4, though this is not apparent from Inati's translation; and in both cases it's clear that Ibn Sīnā allows cancelling the two *laysa*s. The fact that the one statement about Ibn Sīnā's logic in (Marenbon 2012) is dubious speaks volumes about the state of scholarship in this area. We are all suffering from the shortage of commented translations of key Arabic texts.

§8. Translation of Ibn Sīnā, *Qiyās* viii.3 This translation follows the Cairo text (Ibn Sīnā 1964), except for textual emendations recorded in the notes that follow. The paragraph numbering is mine.

viii.3 On the syllogism of absurdity

408.4 [8.3.1] The syllogism of absurdity is really a compound syllogism formed from just
408.5 two propositional syllogisms. Thus, if the goal is a predicative proposition —this is the case which is investigated in the *[Prior] Analytics*—then the conclusion is this predicative [proposition]. But the [compound] syllogism itself will be a propositional one and won't contain a predicative syllogism, at least when it is put in the natural and convenient way. Of the two propositional syllogisms in it, the first is recombinant and has a premise consisting of a meet-like propositional compound with overlapping first and second clauses. The second [of the two propositional syllogisms] is a propositional meet-like duplicative syllogism. In this form the [syllogism of] absurdity is complete—there is no need for any contrived and elaborate explanation, in order to give an analysis of the complete form of the syllogism of
408.10 absurdity, and [to determine] how many syllogisms are needed to complete it. There is no need for the kind of lengthy exposition of [this syllogism] that one finds in the literature.

[8.3.2] The right way to look at it, which is how the the First Teacher approached 408.12
it, is as follows. Suppose for example that we take the goal to be

(37) Not every C is a B.

Now we say:

(38) If the sentence 'Not every C is a B' is false, then every C is a B.

Then we add to it a true premise:

(39) Every B is an A.

We have here one of the recombinant syllogisms that we counted as propositional,
yielding 408.15

(40) If the sentence 'Not every C is a B' is false then every C is an A.

Then we say: 409.1

(41) But not every C is an A.

and in this way we get an impossible absurdity. This duplicates the contradictory negation of the second clause [of (40], yielding the contradictory negation of the first clause [of 40], namely

(42) Not every C is a B.

This is plain sailing.

[8.3.3] This compound syllogism in its complete form consists of two syllogisms. 409.3
Each of the syllogisms has a premise that is a propositional compound. The first of these two [premises] takes the same form regardless of the topic, in the sense that its
first clause expresses that the goal is false and its second clause is the contradictory 409.5
negation of the goal. In the second [of these two premises], the first clause always takes the same form, but the form of the second clause varies. In fact its first clause expresses that the goal is false. But its second clause takes whatever form follows if we take as premise pair the contradictory negation of the goal and the [given] true premise. One kind of premise pair yields a predicative conclusion; [this is used] when the goal is predicative. Alternatively the premise pair yields a propositional compound, when the goal was a propositional compound.

[8.3.4] As we said after the claim, it goes like this: 409.9

(43) If it is not the case that when ϕ then ψ, then it is not the case that whenever ϕ then ψ.

and: 409.10

(44) Whenever χ then ψ.

This yields:

(45) If it is not the case that when ϕ then ψ, then it is not the case that when ϕ then χ.

But this, namely

(46) It is not the case that whenever ϕ then χ.

gives an absurdity. This yields;

(47) Whenever ϕ then ψ.

409.13 This is how to analyse the syllogism known as 'by absurdity, [arguing] towards its premises'.

409.14 [8.3.5] There are people who try to posit the first propositional compound, and
409.15 then prove the absurdity from it, saying 'But its second clause is impossible'. In fact they reckon that

(48) The second clause is impossible.

is what has to be proved. One of them goes to great trouble to find a syllogism which brings together the second clause and impossibility. He says:

(49) The second clause and something true combine to make a syllogism that yields an impossibility; so the conjunction of the second clause and a truth is an impossibility.

Then he produces a syllogism that yields the minor premise, and he says:

(50) The second clause combines with etc. etc. to make a syllogism which yields an impossibility; what we get by combining it with etc. is a syllogism which yields an impossibility. So the second clause combines with a truth to give a syllogism that yields an impossibility.

410.1 This is after unification of the premises has taken place! It takes him a lot of elaborate explanation and lengthy discussion to get to the impossibility.

410.3 [8.3.6] And one of them avoids this. He takes a premise-pair consisting of the second clause and something true, which yields an impossibility. Then he reconsiders and says:

(51) This conclusion is an impossibility, so [the impossibility] comes either from the major premise, or from the minor, or from the premise-pair.

Then he uses a duplicative argument: it doesn't come from the premise-pair, and this yields that it comes either from the major premise or from the minor. Then he uses another duplicative argument: it doesn't come from the major premise, since the major premise is true, so this yields that it comes from the minor premise. Then he says: the minor premise is impossible, and this yields that the contradictory negation of the second clause is true and the contradictory negation of the first clause is true. But all of these kinds of mutilation, and [these] syllogisms that are hidden and not explicit, lengthen the discussion but give us no new information.
410.10 [By contrast] the account we have given is exactly the absurdity syllogism itself, no more and no less.

410.11 [8.3.7] The usual way to use absurdity is to use the recombinant [syllogism], and then you leave its [real] conclusion unmentioned; instead one mentions what is in reality a duplicate of the contradictory negation of its second clause, [and adds]

(52) So this yields the goal.

For example the usual way [to present an argument from absurdity] is to say

(53) If [it's not the case that] not every C is a B, then every C is a B.
But every B is an A, so every C is an A, and this is impossible.
Hence [not] every C is a B.

Thus when one says "so every C is an A", this means 410.15

(54) If [it's not the case that] not every C is a B, then every C is an A.

[In other words,] if the case is as we described, then every C is an A. And the statement "This is impossible" means

(55) Not every C is an A.

—which duplicates the contradictory negation of the second clause [of (54)]. So the usual style agrees with our analysis of the absurdity syllogism.

[8.3.8] The phrase 'syllogism of absurdity' means a syllogism in which the argument reaches an impossibility, so the word 'absurdity' (*ḵalf*) refers to impossibility. 411.1
Some people say that the syllogism of absurdity is called *ḵulf*. These people are out of line; *ḵulf* is just about promises. Also some people have said that it is just called syllogism of *ḵalf* because it approaches the [goal] from behind it (*ḵalfih*) and not through the front door—since it approaches by way of the contradictory negation of the goal. But it seems to me that the most realistic [explanation] is that *ḵalf* is 411.5
used here in the sense of impossibility, not in any other sense.

§9. Notes on *Qiyās* viii.3

[8.3.1]

408.4 'The syllogism of absurdity': Read *wa-qiyāsu l-ḵalfi*. The upper part of the deep forms in (3) above will generally be a recombinant propositional syllogism only in a loose sense. Probably by 'recombinant' here Ibn Sīnā means that the conclusion includes material coming from more than one premise, and 'propositional' means that he regards the displayed $\rightarrow$ as the principal logical operation.

408.7 I have translated *muqaddam* and *tālī* (for p and q respectively in the sentences above) as 'first clause' and 'second clause', because the more usual translations 'antecedent' and 'consequent' have misled too many people—including me in a first version of this paper—into treating all the meet-like sentences as logical 'if-then' sentences, which is certainly wrong for the third and fourth forms in (23) above.

408.8 'with overlapping first and second clauses': The referee remarks that the unpointed manuscript text which the Cairo edition takes as *wa-muqaddamuhu yushārikuhā* could also be read as *wa-muqaddamatin tu-shārikuhā*, and the sense then would be that the first syllogism has a meet-like propositional compound premise and a second premise that overlaps with the second clause of the first premise. Both readings agree with the syllogisms that Ibn Sīnā gives as illustrations. The referee's reading is slightly more fluent Arabic, but it makes Ibn Sīnā point out a less distinctive feature of the syllogism.

408.9 'contrived and elaborate explanation': Read *takalluf* with two mss.

[8.3.2]

408.12 'how the First Teacher approached it': The First Teacher is Aristotle. There is no evidence in Aristotle's text to support Ibn Sīnā's attribution of this view to Aristotle. This is one of a number of places where Ibn Sīnā apparently assumes that Aristotle was such a good logician that he must have shared Ibn Sīnā's own insights.

[8.3.3]

409.2 For *'anna kulla* read *'an laysa kullu*, following ms *sā*.

[8.3.4] The translation uses the following abbreviations: ϕ = 'C is D', ψ = 'H is Z', χ = 'I is U'.

[8.3.7]

409.3f 'syllogisms': Ibn Sīnā is careless about distinguishing simple syllogisms (one step) from compound syllogisms (more than one). From the context it seems that the second syllogism here is simple, but the first could be compound.

410.13 '[it's not the case that]': There are a couple of 'not's missing in all the manuscripts. It could be for example that there should be *mā* before *kāna*, and then *laysa* before *kullu j b*. This is unwelcome evidence of the logical incompetence of some very early copyist.

[8.3.8]

411.3 'promises': Ibn Manẓūr's lexicon *Lisān al-ᶜarab* sv. *ḵlf* explains *ḵulf* as a verbal noun from the verb *'aḵlafa* 'to default (on a promise)'; this verb occurs in the *Qur'ān* at *Sūrat Ṭā-Hā* 86f, which Ibn Sīnā probably has in mind. See (Zimmermann 1981) p. 198 footnote 6 for evidence that the reductio syllogism was vowelled *ḵalf* in the 10th century.

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