Physical metaphors for deduction

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`http://wilfridhodges.co.uk/arabic53.pdf`

Avicenna, c. 980–1037

Ibn Sīnā (= Avicenna, 11th century Persia) developed his own form of logic, intermediate between Aristotle's non-modal syllogisms and modern many-sorted first-order logic.

For example Aristotle's sentences had only one quantifier,
but Ibn Sīnā's sentences have more than one quantifier,
and the quantifiers can be universal or existential independently.
Also Ibn Sīnā's sentences are sortal with two sorts,
object and *time*.

Following Oscar Mitchell (1880s) we call this kind of logic
two-dimensional logic or *2D logic*.
Details at `wilfridhodges.co.uk/arabic44.pdf`,
cited below as *Background*.

Like Ibn Sīnā himself, we can use his 2D logic as a test-bed for theories about the nature of deduction.

Ibn Sīnā was interested in justifying 2D logic over against Aristotle's,
and in seeing how 2D logic too could be extended.

Besides the logical facts themselves,
the background data that Ibn Sīnā invoked included

(1) our reasons for having a logical theory, and
(2) the human mind/brain as a deduction engine operating in space.

The *bāl* or deduction engine

An old Arabic metaphor ‘It came into my *bāl* that . . . ’
means just ‘It occurred to me that . . . ’.
Ibn Sīnā takes the metaphor and turns it into a technical expression for what happens when the human mind performs logical operations.

He thinks of the premises of an argument as being fed into the *bāl* in linear order and processed there.
The *bāl* is something between a mixing bowl and a Turing machine.

He makes two main claims about the operation of the *bāl*:

The Right Order Claim. If the sentences don’t come into the *bāl* in the right order (*tartīb*), no deduction will take place.

The Unification Claim. Two adjacent sentences interact in the *bāl* through having the same expression at their adjacent ends.

The Right Order Claim

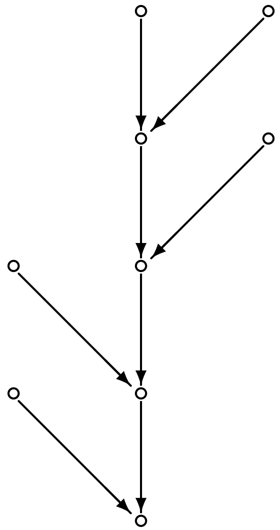
A key point for Ibn Sīnā is that in logical analysis we need to arrange the premises in the right order.
Some later Arabic thinkers picked this up from him and regarded ‘putting in logical order’ as the main contribution of logic.

He says in several places that for debating it’s useful to be able to present the premises in the *wrong* order.
Then one’s opponent may be induced to accept propositions without realising until too late what he’s committed himself to.
(The idea is already in Aristotle, but Ibn Sīnā explains it in terms of the *bāl*.)

But what is the logical order? Is it the order in which the premises are presented, or the order in which they are processed in the deduction?

For Aristotelian non-modal syllogisms with any number of premises, the two questions needn’t be distinguished.

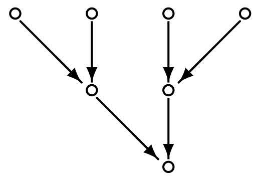
The deduction can be arranged with a linear backbone;
each premise is introduced just once,
and is immediately combined with a sentence in the backbone so as to produce the next sentence in the backbone.
In fact they can be introduced from the same side, except perhaps for a single switch of side.



Aristotle *De Anima* i.3: ‘Demonstrations … don’t go in a circle. They advance in a straight line by addition of terms.’

This is proved in *Background* on the basis of Ibn Sīnā’s analysis of compound syllogisms.
Ibn Sīnā points out that it is not obvious;
two things could go wrong.

First, two premises could interact on the side, before being connected with the rest of the demonstration.



Second, more seriously, a premise could be needed twice or more in the same demonstration.
There are examples in 2D logic, though I don’t think Ibn Sīnā realised this, and it took me several months to find the following example. (Ibn Sīnā assumes a thing can’t be an A or a B at times when it doesn’t exist.)

> Something that is sometimes a B is an A all the time it’s a B.
> Everything that is sometimes an A is a B all the time it exists.
> Therefore something that is sometimes a B is an A all the time it exists.

Ibn Sīnā’s own example to make the point is Proposition 1 of Euclid’s *Elements*. He says he gives this example to explain the notion of *tartīb*.
He numbers the inference steps. Their conclusions:

(1) $AB = AC$.

(2) $AB = BC$.

(3) $AC = BC$ by (1), (2).

(4) ABC is equilateral, by (1), (2), (3).

The steps with conclusions (1) and (2) are independent, and both have to be made before (3).
Also the conclusions of (1) and (2) are each used in two places, as premises for (3) and for (4).

Ibn Sīnā's Euclid example shows (at least prima facie) that the straight-line model of deduction won't work.
This is a challenge to a traditional assumption voiced by Descartes, Rule 3 for the Ordering of the Mind.

> Many things are ... deduced from true and known principles by the *continuous and uninterrupted action* of a mind that has a clear vision of each step ... we remember that we have taken [the steps] successively under review and that *each single one is united to its neighbour*.

My italics. None of the commentators on this passage in the *Cambridge Companion to Descartes* show any awareness of the difficulties.

The statements above about categorical syllogisms refer to their normal use in validating arguments, where all valid syllogisms can count as inference rules.
But Ibn Sīnā agreed with Aristotle that some of these moods are not self-evident, so they need to be justified by deeper methods using first-figure syllogisms, conversions etc. as axioms.

At this deeper level, the linearity properties mentioned above can fail.
Ibn Sīnā's analysis shows that there is also a new problem here.

Aristotle justifies some of the non-self-evident syllogisms using reductio ad absurdum.
Reductio ad absurdum looks to be not straight-line.
Ibn Sīnā has a way of making it straight-line (close to Frege's), but using propositional logic which in turn needs justification.

So Ibn Sīnā must find justifications not using reductio ad absurdum.
These can be found for all moods except *Baroco*;
Philoponus (6th century) said it's impossible for this mood.

But Ibn Sīnā finds a method for *Baroco*.

We will see that both Philoponus and Ibn Sīnā are right, depending on what you want from a justification.

Baroco as Ibn Sīnā (and probably Philoponus too) understood it:

> Either some C is not a B, or there are no Cs.
> Every A is a B, and there are some A.
> Therefore either some C is not an A, or there are no Cs.

Ibn Sīnā introduces a new term D meaning 'a C but not a B'.
He replaces the first premise by two sentences:

Every D is a C, and there are some Ds.
No B is a D.

Since every A is a B and no B is a D, no A is a D (by axiom)
so no D is an A (by axiom).
So some Cs (namely the Ds) are not As.

Snag! ...

The snag is that 'There are some Ds' doesn't follow from the first premise, because of the clause 'or there are no Cs'.
Probably this is why Philoponus ruled out this route.

Nevertheless the derivation is justified by the fact that both the first premise, and Ibn Sīnā's replacement for it, *imply the same Aristotelian sentences*.
(Details given in *Background*, in terms of notions used by Ibn Sīnā.)

So the step is not from a proposition to another proposition that it entails (as in Descartes' picture).
Instead the justification is procedural or engineering.

This example might be a local problem about Ibn Sīnā's logic.
But the same problem appears in natural deduction or Hilbert-style systems,

$$\frac{\phi(c)}{\forall x \phi(x)}$$

when c doesn't occur in any assumptions on which the first line depends.

We can justify these situations by a general assumption that everything takes place 'inside a disjunction' or 'inside a universal quantification'.
But these justifications fall foul of Ibn Sīnā's Unification Claim, to which we turn.

Recall: **the Unification Claim** is that two adjacent sentences interact in the *bāl* through having the same expression at their adjacent ends.

Ibn Sīnā calls the interaction *idġām*, a linguistic term for how the sound at the beginning of a word interacts with a related sound at the end of the previous word.

The heart of the claim seems to be that logical processing doesn't ever go beyond the syntactic top level of the sentences involved. This is true of categorical syllogisms, and is sometimes called *top-level processing*.

Around 1200 the Arabic linguist Al-Sakkākī read some logic in order to make comments from the point of view of linguistics, and came up with the following tautology:

$$((p \leftrightarrow q) \rightarrow$$
$$(((p \rightarrow q) \rightarrow (q \rightarrow p)) \rightarrow ((\neg p \rightarrow \neg q) \rightarrow (\neg q \rightarrow \neg p))))$$

He asks no questions, but one question is obvious:

How the hell do you prove this tautology using only the top-level methods of traditional logic?

Ibn Sīnā's propositional logic has a method that works for this, but only at the cost of abandoning the Unification Claim.

Ibn Sīnā never resolves the problem. But it seems from modern examples that we have two options:

(a) Apply rules at a deep syntactic level.
(b) Remove parts of the sentence before applying the deduction, and then restore afterwards.

I think neither (a) nor (b) makes sense in Descartes' picture.
In practice modern logicians tend to use (a) for substitution of identicals, and (b) everywhere else.

Substitution of identicals

$$\frac{\phi(s), \quad s = t}{\phi(t)}$$

Leibniz stated the rule, but Boole 1847 seems to have been the first to state explicitly that it applies at arbitrary syntactic depth in ϕ.

It involves syntactic analysis of ϕ, not included in Descartes' picture.

Frege attempts to generalise to applications of modus ponens at certain positive sites in a formula:

$$\frac{\phi(\psi), \quad (\psi \rightarrow \chi)}{\phi(\chi)}$$

where ψ is positive in ϕ.

He sets up the notation of *Begriffsschrift* to make it obvious when ϕ has the form

$$(\theta_1 \rightarrow (\theta_2 \rightarrow \ldots \rightarrow (\theta_n \rightarrow \psi) \ldots).$$

In fact this may have been one of the main aims in his design of this notation.
Today nobody even notices this.

Remove and restore

This is the norm in natural deduction.
Ibn Sīnā and Burley (14th c.) both suggested examples,
but neither of them pursued the idea of iterating this approach.

Again this makes no sense in Descartes' picture.
There can be an arbitrarily great distance in the demonstration
between the removal of a piece and its restoration.

Wilfrid Hodges, 'Traditional logic, modern logic and natural language', *Journal of Philosophical Logic* 38 (2009) 589–606.

Wilfrid Hodges, 'How Boole broke through the top syntactic level', in *Proceedings of the History of Modern Algebra: 19th Century and Later Conference*, ed. H. Charalambous et al., Aristotle University of Thessaloniki, Thessaloniki 2010, pp. 73–81.

Wilfrid Hodges, 'Ibn Sīnā on reductio ad absurdum', *Review of Symbolic Logic* (to appear).

Wilfrid Hodges, *Mathematical Background to the Logic of Ibn Sīnā*, draft of book submitted, `http://wilfridhodges.co.uk/arabic44.pdf`.