How far did Avicenna get with propositional logic?

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http://wilfridhodges.co.uk/arabic58.pdf

Avicenna (c. 980–1037, active in Persia) had several logical systems that modern scholars count as 'propositional logic'. We will describe three central ones, which we call PL1, PL2 and PL3.

Arabic logicians described them as *sharṭī*, literally 'conditional'. But they are not just about conditional sentences, so Avicenna suggests *sharṭī* should be understood as meaning sentences with subclauses which are not asserted when the sentence as a whole is asserted.

E.g. 'When the sun is up it's daytime'.

PL1

This is what was available of propositional logic before Avicenna. His predecessor Al-Fārābī (c. 870–c. 950) expounds it. It's an amalgam of ideas from Aristotle's Greek successors (e.g. Theophrastus) and Stoic logicians, with other elements from unidentified sources.

This logic classifies binary sentence operators $\star$ that obey rules of inference like

$$\begin{array}{lll} p\star q, & p & \vdash & q. \\ p\star q, & p & \vdash & \neg q. \\ p\star q, & \neg q & \vdash & p. \end{array}$$

Etc. There is almost nothing beyond the classification.

(1): $p\star q,\ p\ \vdash q$.
$\star$ is said to be 'continuous'. E.g. 'If p then q'.
(This gives us modus ponens.)

(2): (1) together with $p\star q,\ q\ \vdash p$.
$\star$ is said to be 'continuous complete'.
E.g. 'p if and only if q'.

(3): $p\star q,\ p\ \vdash \neg q$, and $p\star q,\ q\ \vdash\ \neg p$.
$\star$ is said to be 'separated'.

(4): (3) together with $p\star q,\ \neg p\ \vdash q$, and $p\star q,\ \neg q\ \vdash\ p$.
$\star$ is said to be 'strict separated'.

(3) and (4) were taken to be different readings of 'Either p or q'.

PL2

This logic, which Avicenna invented, has four sentence forms:

(a) Whenever p then q.

(e) Whenever p then not q.

(i) Sometimes p and q.

(o) Sometimes p and not q.

These are read as obeying the same logical laws as

(a) Every case of p is a case of q.

(e) No case of p is a case of q.

(i) Some case of p is a case of q.

(o) Not every case of p is a case of q.

These laws were already known from Aristotle's syllogistic.

Remark 1.

'Whenever' is *kullamā*. Earlier logicians had used 'if', *in*.
In Arabic grammar, *in* and *kullamā* are very different beasts.
'*in* p, q' is nonextensional; for example it is often taken to represent uncertainty about whether or not p.
'*kullamā* p, q' is purely extensional and represents a quantification over times.

Avicenna's move from *in* to *kullamā* is one illustration of his broader project to rebuild logic on extensional foundations.

Remark 2.

Aristotle's categorical logic proceeds as if 'Every A is a B' is understood as implying that there is at least one A.
This 'existential import' was pointed out by al-Fārābī, and endorsed by Ibn Sīnā with a supporting argument.

So for example we have the implication

Whenever p then q.
Therefore: Sometimes q and p.

This makes 'Whenever p then q' useless for modus ponens applied to a sentence p which is never true.
So PL2 is neat, but inadequate for propositional reasoning.

Remark 3.

In the terminology of PL1, 'Whenever p then q' is a continuous sentence.
So one achievement of PL2 is to extend 'continuous logic' by adding three new sentence forms.
A second achievement is to reduce the formalism of this 'continuous logic' to that of Aristotle's categorical syllogisms.
This is not the only place where Avicenna points out that one formalism can be applied to two different logical questions.

A logic similar to PL2 was invented independently by the mathematician John Wallis in his *Institutio Logicae* 1687.

PL3

This logic was an attempt to extend the extensionalisation of continuous logic to include separated logic as well. Avicenna takes as his starting point the form

(a) Always either not p or not q.

The class of sentence forms must be closed under negation, so we also have

(o) Sometimes both p and q.

Now he has the problem of extending the sentence forms to include corresponding sentences for (e) and (i). Avicenna's calculations indicate two different extensions:

Version A:

(a) Always either not p or not q.
(e) Never both not p and q.
(i) Sometimes both not p and q.
(o) Sometimes both p and q.

Version B:

(a) Always either not p or not q.
(e) Never both p and not q.
(i) Sometimes both p and not q.
(o) Sometimes both p and q.

Avicenna gives a large number of detailed calculations, and we can see which version he is using. He sometimes uses one and sometimes the other, though he never muddles up the two. A clear majority of cases fit Version A and not Version B.

A case can be made that in fact he intended

(e): p and q never have different truth-values.

But he found this too complicated to handle, so he allowed the user to choose between Version A and Version B as appropriate. This is not a stable position to take, but perhaps he saw no way out of it. Henceforth we stick with Version A.

Note that in Version A both (e) and (i) have a negation on the first part, p. In Aristotelian categorical logic there is no sentence form where the first part (the 'subject') is negated. In an Aristotelian sentence with negated first part the negation has to be treated as 'metathetic', i.e. not part of the sentence form, and hence invisible to logic.

Version A is therefore catastrophic for Aristotelian logic. It destroys the distinction between negation 'in the form' and metathetic negation. In PL3 we can put negations wherever we want to. Also Avicenna saw that this destroys his motivation for existential import, so in PL3 it is ignored (and so we have modus ponens back again).

Thus in PL3 we can express all of the forms

$$\forall t\,(p(t) \to q(t)),$$
$$\forall t\,(\neg p(t) \to q(t)),$$
$$\forall t\,(p(t) \to \neg q(t)),$$
$$\forall t\,(\neg p(t) \to \neg q(t))$$

together with the negations of these formulas.
In effect we have a formalism allowing us to express

$$p \subseteq q, \quad \overline{p} \subseteq q, \quad p \subseteq \overline{q}, \quad \overline{p} \subseteq \overline{q},$$
$$p \cap \overline{q} \neq \emptyset, \quad \overline{p} \cap \overline{q} \neq \emptyset, \quad p \cap q \neq \emptyset, \quad \overline{p} \cap q \neq \emptyset.$$

Avicenna could prove all the logical relations between these expressions. In particular he proved three 'syllogisms' not expressible in Aristotelian logic even with metathesis, for example the 'propositional' version of

No C is a B.
Some B is not an A.
Therefore some non-C is not an A.

At least one of these three new 'syllogisms' was found independently by Pseudo-Scotus in his *Questions on the Prior Analytics*, probably 14th c.

Another part of Avicenna's propositional logic, not incorporated with PL1, PL2 or PL3, allows him to use sentences $\phi \to \psi$ both as premises and as conclusions. (Cf. Hodges, 'Ibn Sina on Reductio ad Absurdum', *Review of Symbolic Logic* online.)

If he had combined this with PL3 he would have been in a position to write down and apply all the axioms of boolean algebra.

In fact the first formal system adequate for this purpose was given by Leibniz in the late 17th century.
Cf. Malink and Vasudevan, *Review of Symbolic Logic* 9 (2016) 686–751. (But note that having tools adequate to do X is not the same as doing X.)

Details and references are given at

`http://wilfridhodges.co.uk/arabic44.pdf`

I'm also aiming to get a paper on Avicenna's propositional sentence forms written before the end of the year.