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12th century Arabic logic diagrams: practice and theory

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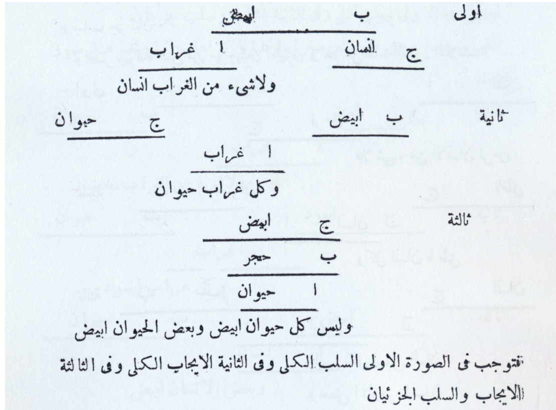
Abū al-Barakāt bin Malka al-Baghdādī



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Abū al-Barakāt lived c. 1085–c. 1170.
He was a Baghdad Jew who converted to Islam.
His main book: *Kitāb al-Muʿtabar*, 'Book of things I considered'.
It's an encyclopedia, using Ibn Sīnā's *Shifā'* (c. 1024) as template.
It contains the earliest statement that bodies fall with constant acceleration.

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Aristotelian logic

Compared with modern logic:
1. The formulas are different.
2. The questions asked are different.
3. Often we have to work out what the earlier logicians meant from what they *do*,
because what they *say* doesn't answer our questions.

Aristotle's categorical sentences (as understood by al-Fārābī and Ibn Sīnā, some reckon by Aristotle too):

Every C is an A.	No C is an A.
$\forall x(Cx \to Ax) \wedge \exists x Cx$	$\forall x(Cx \to \neg Ax)$
$\Downarrow$ *contradictions*	$\Downarrow$
Some C is an A.	Not every C is an A.
$\exists x(Cx \wedge Ax)$	$\exists x(Cx \wedge \neg Ax) \vee \forall x \neg Cx$

A 'formal sentence' has letters as well as words.
An 'interpretation' is a list of letters, that assigns to each letter a singular noun phrase.

Given a formal sentence ϕ and an interpretation I, we write $\phi[I]$ for the sentence got from ϕ by replacing each of the letters of ϕ by the phrase assigned to it by I.

We say that I 'verifies' ϕ, and is a 'model' of ϕ, if $\phi[I]$ is true.

Aristotle considers premise-pairs consisting of two categorical sentences with one letter in common.
Each premise-pair has certain formal sentences as 'candidate conclusions'.

The candidate conclusions for a categorical premise-pair are the categorical sentences whose first/second letters are the letters that are just in the first/second premise.

A premise-pair is 'productive' if one of the candidate conclusions is a logical consequence of it. Then the strongest such candidate is the 'conclusion' of the premise-pair.

A premise-pair is 'nonproductive' (Ibn Sīnā says 'sterile') if none of the candidate conclusions is a logical consequence of the premises.

E.g.

No C is a B. Some B is an A.

is sterile. It has logical consequence 'Some A is not a C', but the letters are the wrong way round.

The central question: Which premise-pairs are productive, and with what conclusions?

Aristotle: (1) To show that θ is a *conclusion* of Φ, we give a proof of θ from Φ in a formal proof system.

(2) To show that Φ is *nonproductive*, we give two models I, J of the premises Φ, such that I verifies 'Every C is an A' and J verifies 'No C is an A'.
Then I shows that Φ doesn't have a negative conclusion, and J shows that Φ doesn't have an affirmative conclusion.
So together I and J show that Φ is sterile.

Abū al-Barakāt runs through all the 48 categorical premise-pairs, and for each one he either gives a conclusion or shows that it's sterile.
But not by Aristotle's methods.
In each case Abū al-Barakāt gives between two and four interpretations,
and for the sterile cases always *three* interpretations.
Also his interpretations are always *nonempty* in the sense that the listed noun phrases always describe nonempty classes; this was new.

What is he doing?

Nonempty classes

Requiring nonempty classes doesn't change the logical relationships between the categorical sentences,
or which premise-pairs are productive.
But it allows simpler formulas for expressing them.

Every C is an A.	$\forall x(Cx \to Ax)$
Not every C is an A.	$\exists x(Cx \wedge \neg Ax)$

The others stay the same.

Nonproductive premise-pairs

There are three types of interpretation, say of C and A:

Type One verifies 'Every C is an A'.
Type Two verifies 'No C is an A'.
Type Three verifies 'Some but not every C is an A'.

Abū al-Barakāt gives for each sterile premise-pair three models, one of each of the three types for C and A.
(Aristotle gave just one of Type One and one of Type Two.)

A theorem that explains what Abū al-Barakāt is doing:

Theorem

Suppose we use only nonempty interpretations, and let Φ be a premise-pair. Then the following are equivalent:

(a) *Φ is sterile.*

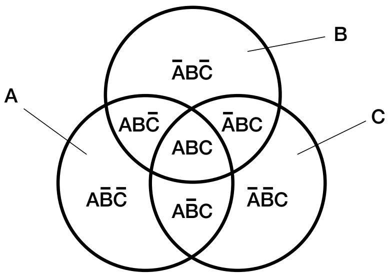
(b) *Φ has models of all three types.*

Productive premise-pairs

Since Abū al-Barakāt gives only interpretations, not proofs, he must be using a model-theoretic notion of logical consequence (cf. Tarski 1936 on logical consequence).
There is essentially only one such notion:

> *θ is a logical consequence of Φ if and only if every model of Φ is a model of θ.*

For Abū al-Barakāt, assume that models are nonempty.
Note the 'every': we quantify over *all* (of infinitely many) models of Φ. How?



Here $\overline{A}$ means 'not-A', and likewise $\overline{B}$, $\overline{C}$.
What categorical sentences are true depends only on which of the seven labelled areas are empty.

So we only need to consider $2^7 = 128$ interpretations.
Still too many for practical proofs.

But we can ignore the empty interpretations:

All classes empty	*1*
Two classes empty	*3*
One class empty	*15*
Total empty structures	*19*

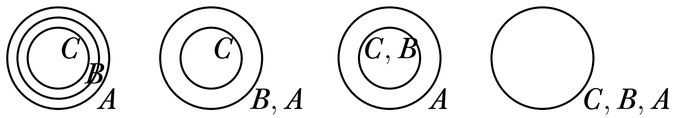
This brings down to 128 - 19 = 109. Still too many.

In practice, limiting to models of Φ cuts down a lot, particularly when Φ is productive.

Example:

Every C is a B. Every B is an A.

Four possibilities according as $C \subset$ or $= B$, and $B \subset$ or $= A$:



Abū al-Barakāt gives interpretations for exactly these four cases, and shows that all these interpretations verify 'Every C is an A'.

For 'Every C is a B. No B is an A' there are just two cases, and Abū al-Barakāt gives exactly these.

For 'Some C is a B. Every B is an A' there are sixteen cases, and Abū al-Barakāt gives just four of them.

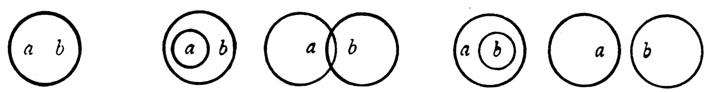
So we can see what Abū al-Barakāt is doing.

He is replacing all proofs by searches through the space of models.

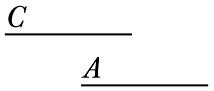
His procedure is correct, and it constitutes a decision procedure for productivity.
When there are more than four models,
in the *productive* case he limits himself to a sample for illustration,
and in the *nonproductive* case he chooses three as examples of all three types.

The pictures

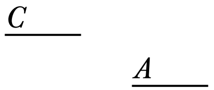
Since he is listing models, his pictures represent models and not sentences.
This makes them different from the diagrams of Euler and Venn, and more like those of Gergonne (1816/7):



Samples:



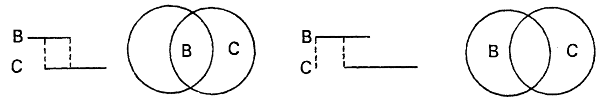
represents that C straddles A.



represents that C is disjoint from A.

Since classes are represented by line segments, there is no way of representing empty classes. So Abū al-Barakāt excludes empty classes.

The circle diagrams of Gergonne, Euler and Venn were anticipated by line diagrams of Leibniz:



Some B is C Some B is not C

cf. Margaret Baron 1969. Baron points out that Ramon Llull already had some kind of circle diagram in the 13th century, though for religious propaganda rather than logic.
But Abū al-Barakāt was already using his line diagrams to do logic in the twelfth century.

Was this shift from proofs to models new with Abū al-Barakāt?

Not entirely. Aristotle's method for proving nonproductivity can be seen as model-theoretic, though Aristotle himself reduced it to manipulation of sentences.

A likely catalyst was Ibn Sīnā's attempt to apply Aristotle's method to a new logic invented by Ibn Sīnā. This is a form of propositional logic or boolean algebra, known as PL3.

PL3 works with eight sentence forms, using $\overline{C}$ for complement of C:

$$C \subseteq A, \quad C \subseteq \overline{A}, \quad \overline{C} \subseteq A, \quad \overline{C} \subseteq \overline{A},$$

$$C \nsubseteq A, \quad C \nsubseteq \overline{A}, \quad \overline{C} \nsubseteq A, \quad \overline{C} \nsubseteq \overline{A}$$

Ibn Sīnā checks productivity and sterility of premise-pairs.

Aristotle proved nonproductivity by giving two models that verify two particular sentences.
For PL3 the corresponding method would need *eight* sentences, not just those two.

Nevertheless Ibn Sīnā proves sterility by giving just two interpretations.
So he is not using Aristotle's method.

Instead he is giving one interpretation that falsifies four of the candidates,
and a second interpretation that falsifies the other four.
(Why should that be possible?! But it works.)

Example: For the sterile premise-pair

No C is a B. Every B is an A.

he gives the two interpretations

I: *C = human, B = stone, A = mineral.*
J: *C = human, B = stone, A = bodily object.*

The sentence $\overline{C} \subseteq \overline{A}$, i.e. $A \subseteq C$, must be true in one of the interpretations I, J and false in the other.

So one of I and J must have its universe restricted to humans.
Ibn Sīnā doesn't mention restricting the universe,
here or in the many other cases where it is needed.
His examples generally work with a restriction of universe,
but not without one.

This is close to the context in which De Morgan in 1846 introduced the notion of a 'universe', one of the main ancestors of the notion of the domain of a structure.

So we see Ibn Sīnā in around 1024 probably using full-blooded model theory to prove sterility theorems.
This anticipates Hilbert's use of models in *Foundations of Geometry* (1899).

These facts come to light only by examining what Ibn Sīnā and Abū al-Barakāt actually did in logic,
not just what they said they were doing.
Understandable, but it makes the historian's task trickier.

Margaret E. Baron, 'A note on the historical development of logic diagrams: Leibniz, Euler and Venn', *The Mathematical Gazette* 53 (384) (1969) 113–125.

J. A. Faris, 'The Gergonne relations', *Journal of Symbolic Logic* 20 (3) (1955) 207–231.

Oliver Gutman, *Pseudo-Avicenna. Liber Celi et Mundi*, Brill, Leiden 2003.

Moshe M. Pavlov, *Abu 'l-Barakat al-Baghdadi's Metaphysical Philosophy, The Kitab al-Mu*[c]*tabar*, Routledge, Abingdon 2017.