An Introduction to Arabic Hypothetical Logic

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Content

Introduction

- The hypothetical logic is a kind of propositional logic, containing conditional plus disjunctive propositions.
- In the Arabic systems, this logic is more or less developed, depending on the authors.
- In al-Farabi's system, the basic laws and rules are provided, and some variants are added.
- But Averroes seems to give less importance to this kind of logic, which he considers as secondary.
- Avicenna by contrast, gives it much importance and constracts a very original system of hiw own.
- In later times, it seems that the hypothetical logic is more developed and studied in Western areas (essentially North Africa) than in Eastern areas (Persia and Asia in general).
- In what follows, I will present the main results and the evolution of hypothetical logic in the Arabic tradition(s).

Introduction

- The main problems that I raise are the following:
- What is the hypothetical logic about?
- How is it viewed by Arabic logicians?
- What are the main results gathered by these logicians?
- What are the main advances in this field?
- How can we interpret these results in the light of modern logic?
- In what follows, I will show that this specific system has given rise to many advances both in propositional logic and in predicate logic, mainly with Avicenna's system, but also with some later ones.
- The hypothetical logic studies the propositions called 'hypothetical'.
- These propositions are either conditional or disjunctive.
- Unlike the categorical ones, they contain the logical connectives 'if…then' or 'either…or', which relate two elementary propositions.

1. The different kinds of hypothetical propositions

- In this respect, it is a kind of propositional logic, unlike syllogistic, which is a predicate logic containing quantified propositions.
- However, Avicenna introduces quantification in some of his hypothetical systems and gives it much importance.
- So there is no radical separation between syllogistic and hypothetical logic in his frame.
- The hypothetical propositions are the following:

1. The conditional ones, containing ‘if…then’: If A is B, then C is D
2. The disjunctive ones, containing ‘either…or’: Either A is B or C is D

- The problem is then the following: How to interpret these two kinds of propositions?
- In other words, how the conditional and the disjunction are they defined by the different Arabic logicians?
- We will answer this question by relying on the definitions and the rules provided in each system.

2. The basic moods and their variants in al-Fārābī's frame

- In al-Fārābī's frame, the hypothetical moods presented are first:

  - the *Modus Ponendo Ponens* [**If p then q, but p; therefore q**] (al-Qiyās, p.)

  - the *Modus Tollendo Tollens* [**If p then q, but not q; therefore not p**] (idem, p.)

  - the *Modus Ponendo Tollens* [**Either p or q, but p; therefore not q**] (idem, p.)

  - the *Modus Tollendo Ponens* [**Either p or q, but not p; therefore q**] (idem, p.)
- He also adds some variants, which are the following moods:

  I/ **If not p then not q, but not p; therefore not q (***al-Qiyas assaghīr***,** p. 167)

  II/ **If not p then q, but not p; therefore q** (*al-Qiyas assaghīr***,** p. 167)

  III**/ If not p then not q, but q; therefore p** (*al-Qiyas assaghīr***,** p. 167)

  IV/ **Not (p and q), but p; therefore not q**

  V/ **If p then q or r, but not q and not r; therefore not p** (*al-Qiyas assaghīr***,** p. 167)

  VI/ **P or Q or R , but P; therefore not Q and not R (***al-Qiyās***,** p. 139)

2. The basic moods and their variants in al-Fārābī's frame

VII/ - P or Q or R

- But not q
- Therefore P or R
- But not P
- Therefore R (*al Qiyās*, p. 139)

- However he says that from the following two premises :

"- Not (P and Q)

- But not P

we **can not** deduce:

- Therefore Q" (*al Qiyās*, p. 139, symbols added)

Because Q could be false, since the disjunction is not exhaustive.

- In *al-Maqūlāt* (*Categories*), al-Fārābī says that the conditional can also be "convertible" (p. 79) in some cases and when it is "essential", which gives rise to the following moods:

- 'If p then q; but q; therefore p'
- 'If p then q; but not p; therefore not q' (*al-Maqūlāt*, p. 79)

2. The basic moods and their variants in al-Fārābī's frame

- In this case, we have a double conditional meaning 'if and only if'.
- The essential conditional involves a necessary relation between the antecedent and the consequent. So it is a strict implication ('necessarily if p then q').
- So al-Fārābī's conditional is different from the Stoïc one, which is a material conditional as shown by Lukasiewicz.
- The bi-conditional is called a 'complete' (*tāmm*) implication as witnessed by the following quotation:

"And those expressing a complete implication are those where if *whatever* element holds, the other one necessarily holds too by means of it (*bi-wujūdihi*), for if the first one holds, the second one necessarily holds, and if the second one holds, the first one necessarily holds too." (*al-Qiyās*, p. 127, my emphasis].

- So it is a strict double implication or equivalence.

2. The basic moods and their variants in al-Fārābī's frame

- As to the disjunction, it can be exclusive and exhaustive, as when we say 'either p or q but not both'.
- In this case it is complete and validates both *Modus Ponendo Tollens* and *Modus Tollendo Ponens.*
- But it can also be *only exclusive*, when it validates 'Not (p and q), but p; therefore not q'.
- In this case, it is *incomplete*, because it ***does not*** validate 'Not (p and q), but not p; therefore q'.
- The incomplete disjunction may have an undetermined number of disjuncts, so it is *not exhaustive*.
- But al-Fārābī does not admit the inclusive meaning of the disjunction (i.e. 'either p or q or both'), since according to him, the disjunction always involves some kind of conflict (ʿ*inād*).
- In all cases, the disjunction is ***not truth-functional***, because its truth value depends mainly on the meanings of its elements, not only on their truth values.
- These elements are always incompatible in the examples provided.

3. The systems of Avicenna

- Avicenna admits several kinds of hypothetical systems.
- The first one is similar to that of al-Fārābī and contains what Avicenna calls 'the *istithnā'ī* (exceptive) syllogisms', which are presented in the very end of *al-Qiyās*.
- The second one contains what Avicenna calls '*iqtirānī*' (conjunctive) syllogisms and it includes three sub-systems:

A/ A system which deals with quantified hypothetical conditional propositions, parallel to the categorical ones.

B/ A system which deals with quantified conditional as well as disjunctive hypothetical propositions.

C/ A system which mixes between quantified conditional and disjunctive propositions *and* categorical ones.

- The first system is comparable to al-Fārābī's one, although there are some differences between them.
- For instance, according to Avicenna, the conditional is not convertible, i.e. is not a biconditional, unlike what al-Fārābī says.

3a. The *istithnā'ī* (exceptive) syllogisms

- But like al-Fārābī, Avicenna admits the following syllogisms in his *istithnā'ī* hypothetical system:

  1- *Modus Ponendo Ponens*

  2- *Modus Tollendo Tollens*

  3- *Modus Ponendo Tollens*

  4- *Modus Tollendo Ponens*
- Like al-Fārābī, he also states the syllogism whose first premise is a negated conjunction:

  5. Not (p and q), but p; therefore not q (*al-Qiyās*, p. 452)
- He also states the following syllogism:

  6. Either not p or not q, but p; therefore not q (*al-Qiyās*, p. 451)
- Here, 'not (p and q)' the first premise of (5) is replaced by 'not p or not q' the first premise of (6).
- Since everything else in the two moods is the same, this could mean that he **holds** implicitly the following **De Morgan Law** : **not (p and q) ≡ (not p or nor q)**
- This also shows that he admits the inclusive meaning of the disjunction, which was not mentioned by al-Fārābī.

3a. The *istithnā'ī* (exceptive) syllogisms

- In *al-Ishārāt wa al-Tanbīhāt*, Avicenna defines the disjunction in three distinct ways:

  (1) '$p \underline{\vee} q$': Either p or q, but not both

  (2) '$\sim (p \wedge q)$': Not both p and q

  (3) '$p \vee q$': Either p or q or both
- In his comment on *al-Ishārāt*, Tūsī explains this as follows: in (1) the disjunction "prevents the conjunction (*jam'*) *and* the vacuity (*khulūw*)" [= it *does not admit* the case where p and q are both true, nor the case where they are both false].
- In (2), it prevents *only* the conjunction [= it *does not admit* the case where p and q are both true].
- In (3), it prevents *only* the vacuity [= it *does not admit* the case where p and q are both false].
- This distinction will be used at length by Avicenna's followers, such as Tūsī, al-Khūnajī and many western (North African) scholars as we will see in section 4.

3a. The *istithnā'ī* (exceptive) syllogisms

- As to the conditional, it is either a *luzūmī* one (where the antecedent ***entails*** the consequent) or an *ittifāqī* one (where the consequent does *not really follow from* the antecedent).
- This also will be used by al-Khūnajī and his followers and defined in terms of the truth conditions of the proposition.
- He also admits the following equivalences between the conditional and the disjunctive propositions:

  - $(P \supset Q) \equiv \sim(P \wedge \sim Q)$ (*al-Qiyās*, p. 280)
  - $(\sim P \supset Q) \equiv (P \vee Q)$ (*al-Qiyās*, p. 244.16)
  - $(P \supset Q) \equiv (\sim P \vee Q)$
- These will also be used by Avicenna himself in his quantified hypothetical system .
- He also admits the principle of *contraposition*, i.e. the following:

  - $(P \supset Q) \equiv (\sim Q \supset \sim P)$

  And consequently the following equivalence:

  - $(P \supset \sim Q) \equiv (Q \supset \sim P)$

  Both will be used and discussed by his followers in eastern and western areas.

3.b The *iqtirānī* (connected) syllogisms

- In this kind of hypothetical systems, Avicenna uses quantified propositions which contain either a conditional or a disjunction.
- These propositions are parallel to the categorical **A**, **E**, **I** and **O** propositions, except that they contain whole propositions instead of a subject and a predicate.
- The quantifiers range over situations or times.
- The conditional hypothetical propositions are expressed as follows:

  - **A**C: Whenever (*kullamā*) A is B then H is Z (*al-Qiyās*, p. 265)
  - **E**C: Never when A is B then H is Z (p. 280)
  - **I**C: It happens that (*qad yakūn*) if every A is B then every H is Z (p. 278)
  - **O**C: Not whenever A is B then C is D.

While the disjunctive hypothetical propositions are expressed as follows:

  - **A**D: Always either A is B or C is D
  - **I**D: It happens that either A is B or C is D
  - **E**D: Never either A is B or C is D
  - **O**D: Not always either A is B or C is D

3.b The *iqtirānī* (connected) syllogisms

- The first natural formalization of the conditional ones is the following:
  - **A**c: $(\forall s)(Ps \supset Qs)$
  - **I**c: $(\exists s)(Ps \wedge Qs)$
  - **E**c = ~**I**c = $\sim(\exists s)(Ps \wedge Qs)$ [= $(\forall s)(Ps \supset \sim Qs)$]
  - **O**c = ~**A**c = $\sim(s)(Ps \supset Qs)$ [= $(\exists s)(Ps \wedge \sim Qs)$] (Chatti, 2016)
- This formalization validates the *principle* of *contraposition* for quantified propositions, expressed as follows:
  - ‘$(\forall s)(Ps \supset Qs) \equiv (\forall s)(\sim Qs \supset \sim Ps)$’
- It validates **E**c conversion and **I**c conversion, which Avicenna holds too:
  - **E**c: $(\forall s)(Ps \supset \sim Qs) \equiv (\forall s)(Qs \supset \sim Ps)$
  - **I**c: $(\exists s)(Ps \wedge Qs) \equiv (\exists s)(Qs \wedge Ps)$
- However, it does *not* validate **A**c conversion, for from ‘$(\forall s)(Ps \supset Qs)$’ we cannot deduce ‘$(\exists s)(Ps \wedge Qs)$’, nor does it validate *subalternation*, *contrariety* and *subcontrariety* which Avicenna holds in this system too.

3.b The *iqtirānī* (connected) syllogisms

- It does not validate *Darapti* and *Felapton* either, which Avicenna does admit in this system too.
- To validate these rules and moods, one has to add an 'existential augment' to the **A**c propositions, which are thus formalized as follows:

  - **A**c: $(\exists s)\ Ps \wedge (\forall s)(Ps \supset Qs)$ (consequently **O**c must be formalized accordingly)
- This validates **A**c-conversion, *Darapti* and *Felapton*, plus all the relations of the square.
- So we must say that **A**c admits two formalizations, which can be used in different occasions:

  1. **A**c with the augment and 2. **A**c without the augment.
- These formalizations are not clearly distinguished but they both can be supported by some evidence in the text
- (1) is supported by the following quotation:

"When we say: 'If A is B, then H is Z', we assume from this (*nūjibu min hāḍha*) that at any time where 'A is B' ***is the case*** and when A is B then H is Z, as if the fact that H is Z follows the fact that A is B, ***in so far as in effect*** A is B (*min hayṭhu huwa kā'inun A [huwa] B*)" (*al-Qiyās*, p. 263.8-9, my emphasis).

3.b The *iqtirānī* (connected) syllogisms

- As to (2), it can be justified by the admission of *contraposition* and the equivalences between some conditional and disjunctive propositions.
- **A**c (without the augment): ‘$(\forall s)(Ps \supset Qs)$’ equals ‘$(\forall s)(\sim Ps \vee Qs)$’, that is, $\mathbf{E}_D$.
- Below, we will see that both **A**c’s are needed and are used in different sub-systems.
- As to the disjunctive propositions, they are formalized as follows by Rescher:

  $\mathbf{A}_D$: $(\forall t)\ (Pt \vee Qt)$

  $\mathbf{E}_D$: $(\forall t) \sim (Pt \vee Qt)$

  $\mathbf{I}_D$: $(\exists t)\ (Pt \vee Qt)$

  $\mathbf{O}_D$: $(\exists t) \sim (Pt \vee Qt)$ (Rescher 1963, p. 233)
- Where the disjunction has a univocal meaning (the exclusive one)
- Unfortunately, this formalization does not validate Avicenna’s moods.

3.b The *iqtirānī* (connected) syllogisms

- An alternative formalization has been suggested by Pr. Wilfrid Hodges, which is the following:

"(a, mn) At all times t, at least one of p and q is true at t. [**=** $(\forall s)(Ps \vee Qs)$]

(e, mn) At all times t, if p is true at t then q is true at t. [= $(\forall s)(Ps \supset Qs)$ = $(\forall s)(\sim Ps \vee Qs)$]

(i, mn) There is a time at which p is true and q is not true.[= $(\exists s)(Ps \wedge \sim Qs)$]

(o, mn) There is a time at which neither p nor q is true" (Hodges, forthcoming, p. 263) [= **O**D: $(\exists s)(\sim Ps \wedge \sim Qs)$]

- However, as Pr. Hodges acknowledges, in some cases, one has to slightly modify these formalizations to validate some moods.
- For one thing, Avicenna himself says that when **A**D is "strict", it must be expressed by an ***exclusive disjunction***, which changes the formalization of **O**D accordingly.
- Second: as Avicenna ***does not*** say, **E**D and **I**D can be formalized in another way in some cases, i.e. as follows:

  ED: $(\forall s)(Ps \vee \sim Qs)$

  ID: $(\exists s)(Ps \wedge \sim Qs)$ (Hodges, forthcoming)

3.b The *iqtirānī* (connected) syllogisms

- As a result, we get the following eight propositions:
  AD1: $(\forall s)(Ps \vee Qs)$
  AD2: $(\forall s)(Ps \veebar Qs)$ (strict disjunction)
  ED1: $(\forall s)(\sim Ps \vee Qs)$
  ED2: $(\forall s)(Ps \vee \sim Qs)$ [= $(\forall s)(\sim Qs \vee Ps)$ = $(\forall s)(Qs \supset Ps)$]
  ID1: $(\exists s)(\sim Ps \wedge Qs)$
  ID2: $(\exists s)(Ps \wedge \sim Qs)$ [= $(\exists s)(\sim Qs \wedge Ps)$]
  OD1: $\sim(\forall s)(Ps \vee Qs)$ (the contradictory negation of **A**D1)
  OD2: $\sim(\forall s)(Ps \veebar Qs)$ (the contradictory negation of **A**D2)
- In all cases, however, Avicenna holds the contradictions **A**D/**O**D and **E**D/**I**D.
- We can try to justify the two interpretations of **I**D as follows: $\mathbf{I}_D$ is a *particular disjunctive proposition.*
- As a particular, it could be true in some cases. So it could be true when one of its elements is true but not the other one.
- But the exact element which is true is ***not necessarily the same one***.

3.b The *iqtirānī* (connected) syllogisms

- This is why we could have the following options:
  1. Either P or Q = Q is true but not P [or else "when not P, then Q"].
  2. Either P or Q = P is true but not Q [or else "when not Q, then P"]
- These readings may be considered as *intuitively* plausible, even if they do not correspond to a *unified formal* interpretation.
- In this interpretation, $\mathbf{I}_D$ would say only *one part* of what $\mathbf{A}_D$ says.
- As to the two distinct $\mathbf{A}_D$'s, they correspond to two meanings of the disjunction:
  1. Exclusive (= P or Q but not both) 2. Inclusive (P or Q or both).
- In both cases, the corresponding $\mathbf{O}_D$'s are their respective contradictories, i.e. respectively: ['$\underline{\vee}$' = P or Q but not both]
  - '$\sim (P \underline{\vee} Q)$' = '$\sim P \underline{\vee} Q$' = '$\sim Q \underline{\vee} P$' = '$P \equiv Q$'
  - '$\sim (P \vee Q)$' = '$\sim P \wedge \sim Q$'
- We will see below that Avicenna uses all these meanings in his systems.

3.b1.The *iqtirānī* syllogisms with conditional propositions

- This first system is exactly parallel to the categorical syllogistic, with only conditional propositions.
- **A**c should be the one with the augment, otherwise, **A**c-conversion does not hold and *Darapti* and *Felapton* do not hold too.
- As an example of the moods held, we can state the following (hypothetical) *Barbara* :

“Whenever A is B, then C is D

Whenever C is D, then H is Z

Therefore Whenever A is B then H is Z” (*al-Qiyās*, p. 296.3-4)

- Likewise for the other first figure moods and the second figure moods.
- In the third figure, *Darapti,* for instance, is stated as follows:

“Whenever C is D, then H is Z

Whenever C is D, then A is B

Therefore It happens that when H is Z, A is B” (*al-Qiyās*, p. 302)

It is valid only when **A**c contains the augment.

3.b2.The *iqtirānī* syllogisms with conditional and disjunctive propositions

- The second system contains conditional plus disjunctive propositions.
- The moods held are different from the ones above, although Avicenna tends to reduce the disjunctive propositions to the conditional ones.
- This system is presented in section VI.2 of *al-Qiyās*, which is subdivided into 10 groups (governed by specific conditions), each of which containing 3 or more sub-groups.

For instance, in Group I (sub-group I-a), we find the following mixed *Barbara*:

"Whenever H is Z, then C is D ($\mathbf{A}_C$)

Always either C is D or A is B ($\mathbf{A}_D$)

Therefore Whenever H is Z, then not A is B ($\mathbf{A}_C$)" ([34], 305.8-10)

- Due to the disjunction, the consequent of the conclusion contains a negative element, but the conclusion itself is considered as affirmative by Avicenna.
- This mood is valid only when the disjunction is exclusive. It is not valid if the disjunction were inclusive.
- Other moods are totally new such as the following **AEE** of the first figure:

"Never if H is Z then C is D ($\mathbf{E}_C$)

Always either C is D or A is B ($\mathbf{A}_D$)

Therefore Never either H is Z or A is B ($\mathbf{E}_D$) (*al-Qiyās*, p. 306.3-5)

3.b2.The *iqtirānī* syllogisms with conditional and disjunctive propositions

- This mood is ***not*** *Celarent*; rather it is an **AEE** mood of the *first* figure, which does not correspond to any known mood of categorical logic or of the hypothetical system containing only conditionals. So it is a *new mood*.
- Another **AEE** mood of the first figure, pertaining to Group II (Sub-Group II-b) is the following:

“Never if H is Z then C is D ($\mathbf{E}_C$)

Always either C is D or not (A is B) ($\mathbf{A}_D$)

Therefore Never if H is Z then A is B ($\mathbf{E}_C$) (*al-Qiyās*, 307.15-17)

- Its counterpart with **O**c propositions is also held by Avicenna (*al-Qiyās*, p. 307)
- Some moods are valid with one interpretation of **I**D and not valid with the other one.
- For instance, a mixed *Disamis* **I**D**A**C**I**D (*al-Qiyās*, p. 309.8-9) of group III-a, where the **I**D premise should be interpreted as ‘P $\wedge$ $\sim$Q’, not as ‘Q and $\sim$P’.
- We find also an **AEE** mood in the third figure, which has the form **A**D**E**C**E**C.(group III)
- Its correspondent with **O**c propositions is valid in Avicenna’s frame, provided Oc contains the augment, i.e. is expressed thus: $\sim(\exists s)Ps \vee \sim(\forall s)(Ps \supset Qs)$ (*al-Qiyās*, p. 310.8-10)

3b3. The *iqtirānī* syllogisms with conditional, disjunctive and categorical propositions

- In his third system, Avicenna considers the moods where the hypothetical conditional and disjunctive propositions are combined with categorical ones.
- These syllogisms are presented in section VI-4 of *al-Qiyās*.
- In these moods, the categorical premise "takes the place of the minor or of the major term" (*al-Qiyās*, p. 325.5).
- The moods are structured as the usual categorical ones but their premises may share just one part of the whole proposition.
- The first mood considered is the following:

"Whenever H is Z, then **Every C is D**

Every D is A

Whenever H is Z, then **Every C is A**" (*al-Qiyās*, 326.3-4) [in bold: **Barbara**)

- Here the term shared by the major premise and the consequent of the minor one is the term D.
- The whole mood is valid as it is stated., without any further premise.
- So we can say that Avicenna's logic is ***monotonic,*** since "Monotony states that… if φ is a consequence of Σ then it is also a consequence of any set containing Σ as a subset" (Strasser &Antonelli, "Non-monotonic logic", SEP, 2019).

3b3. The *iqtirānī* syllogisms with conditional, disjunctive and categorical propositions

- In other cases, the presupposed syllogism is *Celarent*, or *Cesare* for instance.
- In section VI-5, he considers the case where the singular proposition 'H is Z' is the consequent of the premises.
- For instance, the following mood:

"**Every C is B**

Whenever **No B is A** then H is Z

If follows: It happens that when **No C is A** then H is Z" (*al-Qiyās*, 338.3-4)

- However, this 'mood' is not valid as it is stated. It becomes valid only when the presupposed mood (here *Celarent*) is stated explicitly as a further premise.
- In section VI-6, he considers another moods containing disjunctions such as the following:

"Every B is either C or H or Z

Every C and H and Z are A

Therefore Every B is A" (*al-Qiyās*, 350.3-4)

- This is called the "divided syllogism". We can formalize it as follows:

$(x)[Bx \supset (Cx \vee Hx \vee Zx)]$ (first premise)

$(x)[(Cx \vee Hx \vee Zx) \supset Ax]$ (second premise)

$\vdash (x)(Bx \supset Ax)$ (conclusion)

3b3. The *iqtirānī* syllogisms with conditional, disjunctive and categorical propositions

- This formalization shows that Avicenna equates between:

  "$(x)[(Cx \vee Hx \vee Zx) \supset Ax]$" and "$(x)(Cx \supset Ax) \wedge (x)(Hx \supset Ax) \wedge (x)(Zx \supset Ax)$".
- This is remarkable, since this equivalence is the quantified counterpart of the following law of Russell & Whitehead's *Principia Mathematica*:

  '$[(q \vee r) \supset p] \equiv [(q \supset p) \wedge (r \supset p)]$' (proposition *4.77, section A p. 121)
- But it also shows that in this mood, the **A** proposition *should not* contain the augment. As to the disjunction, it should be inclusive.
- However, he also presents the following mood of the third figure, where the **A**c proposition *should contain* the augment, because otherwise, the mood would not be valid:

  "Always either C is B or D is B

  Every C and every D is H

Therefore Some B is H" (*al-Qiyās*, 351.10-11)

- So here too, there is a lack of clarity with regard to the different interpretations of the **A**c propositions.

4. Further developments, al-Khūnajī, Ibn ʿArafa and Sanūsī.

- Al-Khūnajī (d.1286, 685 of Hegira) was one of the followers of Avicenna, although as stressed by El-Rouayheb (2019, p. 44) "he was critical of Avicenna" as well as of "of al-Rāzī".
- However his brief 'handbook' *al-Jumal* seems clearly influenced by Avicenna's thought, although it departs from it on some points.
- As claimed by El-Rouayheb, this author had a great influence on North African scholars, for he says:

"Rather than Kātibī's *Shamsiyya* and Urmawī's *Maṭāliʿ*, Khūnajī's *Jumal* became the standard handbook on advanced logic in the Maghreb, and elicited numerous commentaries by fourteenth- and fifteenth-century North African scholars" (2019, p. 121)

- Furthermore, according to El-Rouayheb, these scholars were more interested by hypothetical logic than their Eastern colleagues.
- This is why I will present the views of two of them, namely Ibn ʿArafa (d.1401, 803 of Hegira) and Muhammed b. Yusuf al-Sanūsī (d. 1490, 895 of Hegira)

4. Further developments, al-Khūnajī

- In hypothetical logic, al-Khūnajī clarifies Avicenna's distinctions.
- According to him, the hypothetical is either connected (*muttasila)* or disconnected (*munfasila*) [= disjunctive].
- The connected one is 1. *implicative* (*luzūmīyya*) or 2. *concordant* (*ittifāqīyya*)
- In the *luzūmīyya* (1) the consequent really ***follows from*** the antecedent.
- In the *ittifāqīyya* (2), there is no such relation for "there is just a concordance in truth (*mujarrad ittifāqun fī al-sidqi*)' (*al-Jumal*, p. 6).
- If so, the *ittifāqīyya* seems to be a conjunction, and its truth conditions seem to be settled: True when both elements are true, False otherwise.
- The disjunctive is of three kinds, as in Avicenna's *al-Ishārāt*:

1. Real (*ḥaqīqīyya*): where the elements are opposed both in truth anf falsity (*taʽānud fī al-sidqi wa al-kadhib maʽan*)

2. Preventive of the conjunction (*māniʽatu al-jamʽ*): where the elements are opposed only in truth (*taʽānud fī al-ṣidqi faqat*)

3. Preventive of the vacuity (*māniʽatu al-khuluw):* the elements are opposed only in falsity (*taʽānud fī al-kadhibi faqat*)

4. Further developments, al-Khūnajī

- However, he adds that (1) is true "when each element is contradictory to the other or equivalent to the contradictory of the other" (*al-Jumal*, p. 6), e.g. even and odd.
- (2) is true "when each element is more specific (*akhaṣṣ*) than the contradictory of the other" (*al-Jumal*, p. 6)
- (3) is true "when each element is more general (*aʿamm*) [than the contradictory of the other]" (*al-Jumal*, p. 6)
- This may suggest that the disjunctions are not t*ruth-functional,* because of *the precise relation between* the elements, which are not *any* elements.
- But he adds in the sequel the *whole set of truth conditions* of both the disjunctions and the conditionals.
- Thus the ***real*** disjunction "is true when *only one* of its elements is true, it is false when both are true and when both are false" (*al-Jumal*, p. 6). So it is truth-functional.
- The second disjunction (*mānīʿatu al-jamʿ*) "is true when both elements are *false* or when one of them is *false*, and it is false when both are *true*" (*al-Jumal*, p. 6). So it is also truth-functional.

4. Further developments, al-Khūnajī

- However when stating the truth conditions of the *third disjunction*, he just says: "the [truth conditions] of the preventive of vacuity are the inverse [of the above] (*bi-al-ʽaksi*)" (idem, p. 6).
- This should mean that this third disjunction is true when both its elements or one of them is *true*, and it is false when both are *false*.
- The truth conditions of the three disjunctions are the following:

(1)	(2)	(3)
$P \veebar Q$	$\sim (P \wedge Q)$	$P \vee Q$
1 0 1	**0 1 1 1**	**1 1 1**
1 1 0	1 1 0 0	1 1 0
0 1 1	1 0 0 1	0 1 1
0 0 0	**1 0 0 0**	**0 0 0**

- So it is clear that all the disjunctions are truth-functional.
- This is a *significant move* in hypothetical logic, since it becomes more formal than it was in al-Fārābī's and Avicenna's frames.

4. Further developments, al-Khūnajī

- As to the connected propositions, their truth conditions are the following: "**The connected is true when both elements are true, or when the consequent alone is true or when both are false**, *and it is false when both are false or one of them is false or both are true if it is implicative*" (*al-Jumal*, p. 6, emphasis added).
- Here there is some ambiguity in the last part of the quotation (in *italics)*, for the conditions stated seem incoherent. So there must be some error in the text (which is not properly edited, as it stands).
- But if we recall that the ***ittifāqī*** proposition had been said to express ***concordance in truth***, the truth conditions of the two kinds of connected propositions should be the following:

$P \supset Q$ (*luzūmī*)	$P \wedge Q$ (*ittifāqī*)
1 1 1	1 1 1
1 0 0	1 0 0
0 1 1	0 0 1
0 1 0	0 0 0

- With these truth conditions, the *ittifāqī* appears to be a conjunction and the *luzūmī* is a conditional, whose truth conditions are *close* to that of the material conditional. But the consequent is *said to follow from* the antecedent.

4. Further developments, al-Khūnajī

- But what is really new (comparing to his predecessors) is expressed as follows:

"And the connected multiplies (*tata'addadu*) by the multiplication of the elements of the consequent, not those of the antecedent, due to the necessity of the implication of the part by the whole, but not conversely. And the disjunctive multiplies by the multiplication of its elements, when it prevents the vacuity but not when it prevents the conjunction" (*al-Jumal*, p. 6)

- By this unique phrase, al-Khūnajī expresses the following laws and non implications:

  - $[P \supset (Q \wedge R)] \rightarrow [(P \supset Q) \wedge (P \supset R)]$ (distributivity of conditional over conjunction)
  - $[P \wedge (Q \wedge R)] \rightarrow [(P \wedge Q) \wedge (P \wedge R)]$
  - $[P \vee (Q \wedge R)] \rightarrow [(P \vee Q) \wedge (P \vee R)]$ (distributivity of disjunction over conjunction)

  But

  - $[(P \wedge Q) \supset R] \nrightarrow [(P \supset R) \wedge (Q \supset R)]$
  - $\sim[P \wedge (Q \wedge R)] \nrightarrow [\sim(P \wedge Q) \wedge \sim(P \wedge R)]$

- This is a ***remarkable advance***, although al-Khūnajī leaves the valid converses of some of these implications unstated, for instance '$[(P \vee Q) \wedge (P \vee R)] \rightarrow [P \vee (Q \wedge R)]$' is not stated by him.
- He does not state either the law of distributivity of conjunction over disjunction, i.e. $[P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$

4. Further developments Ibn ʿArafa and al-Sanūsī

- In addition, he evokes explicitly the following laws: '$(P \wedge Q) \rightarrow P$' and '$(P \wedge Q) \rightarrow Q$'
- As to Ibn ʿArafa, he makes the same distinctions as al-Khūnajī between the different disjunctions and connections.
- But he clarifies the truth conditions of the *luzūmīyya* by saying that it is true in "all cases other than when the antecedent is true while the consequent is false"(p.27). So it has the values of a conditional and is consequently truth-functional.
- As to the *ittifāqīyya*, he says that it is true when its elements are true and false otherwise. So it is clearly a truth-functional conjunction.
- The real disjunction is true when its elements have different values, while the second disjunction is true when its elements are both false or one is true while the other is false. They are false otherwise.
- So he confirms the truth-functionality of the two kinds of connections and of the disjunctions.
- He adds that the *real* disjunction "does not comprise three elements" (*al-Mukhtaṣar*, p. 26), due to the fact that its elements are contradictory to each other, while the preventive of conjunction can "comprise more than two elements" (idem, p. 26)

4. Further developments Ibn ʿArafa and al-Sanūsī

- In his *al-mukhtaṣar fī al-Manṭiq*, he states the same implications as al-Khūnajī and some of the non implications.
- He adds that their negations behave in the inverse way (*wa sawālibuhā ʿalā al-ʿaksi fī dhālika*)(p. 27), for instance, the following (which is valid):

$$\sim [(P \wedge Q) \supset R] \rightarrow [\sim(P \supset R) \wedge \sim(Q \supset R)]$$

- As to al-Sanūsī, he says that the connected propositions may involve the notion of cause but not in the same way for :

1. The antecedent is the cause of the consequent *rationally (*ʿ*aqlī*), e.g. 'If this is a human, it is an animal': here the link is necessary. It is called *luzūmīyya (al-mukhtasar,* p. 25*)*.

2. The antecedent and the consequent are related according to some law (*shar*ʿ*ī*), e.g. 'If it is daytime, then the planets are hidden': here the link is just usual *(*ʿ*ādī*), not really necessary.

3. There is no causal relation at all between the two elements, e.g. 'If the sun rises, men are speaking': this is an *ittifāqiyya*.

In this third kind, the elements only *happen to* be true together.

In other examples such as '[even] if (*law*) this man presents his excuses, I would no forgive him' (p. 24). he word 'if' seems to mean 'even if'. It has thus the truth conditions of a conjunction.

4. Further developments Ibn ʽArafa and al-Sanūsī

- As to al-Sanūsī, he states a great number of implications in his ‘*Sharh Mukhtaṣar fī al-Mantiq*’.
- For apart from the implications and non implications already stated by al-Khūnajī and Ibn ʽArafa, he adds the following:

1. $[(P \wedge Q) \wedge R)] \rightarrow [(P \wedge R) \wedge (Q \wedge R)]$(idem, p. 73, El-Rouayheb, p. 131).
2. $[(P \wedge Q) \vee R)] \rightarrow [(P \vee R) \wedge (Q \vee R)]$(idem, p. 73, El-Rouayheb, p. 131)
3. $\sim[(P \wedge Q) \wedge R)] \not\rightarrow [\sim(P \wedge Q) \wedge \sim(P \wedge R)]$(idem, p. 73, El-Rouayheb p. 131)
4. $\sim[P \supset (Q \wedge R)] \not\rightarrow [\sim(P \supset Q) \wedge \sim(P \supset R)]$ (idem, p. 73, El-Rouayheb, p. 131)
5. $\sim[P \wedge (Q \wedge R)] \not\rightarrow [\sim(P \wedge Q) \wedge \sim(P \wedge R)]$(idem, p. 73, El-Rouayheb, p. 131)
6. $\sim[(P \wedge Q) \vee R)] \not\rightarrow [\sim(P \vee R) \wedge \sim(Q \vee R)]$(idem, p. 73, El-Rouayheb, p. 132)
7. $\sim[P \vee (Q \wedge R)] \not\rightarrow [\sim(P \vee Q) \wedge \sim(P \vee R)]$(idem, p. 73)
8. $(P \vee Q) \rightarrow [(\sim P \supset Q) \wedge (\sim Q \supset P)]$ (idem, p. 75, El-Rouayheb, p. 132)
9. $\sim (P \wedge Q)] \rightarrow [(P \supset \sim Q) \wedge (Q \supset \sim P)]$ (idem, p. 75, El-Rouayheb, p. 132)
10. $[(P \vee Q) \wedge\sim (P \wedge Q)] \rightarrow [(\sim P \supset Q) \wedge (\sim Q \supset P) \wedge (P \supset \sim Q) \wedge (Q \supset \sim P)]$ (idem, p. 75, El-Rouayheb, p. 132).

4. Further developments Ibn ʿArafa and al-Sanūsī

11. (P ⊃ Q) → ~ (P ∧ ~Q) (idem, p. 75, 3.1 El-Rouayheb, p. 132)
12. (P ⊃ Q) → (~P ∨ Q) (idem, p. 75, El-Rouayheb, p. 132)
13. ~(P ∧ Q) → (~P ∨ ~Q) (A part of the first De Morgan law)
14. (P ∨ Q) → ~ (~P ∧ ~Q) (A variant of the second De Morgan Law)

- For he says "and each of the preventive of conjunction and the preventive of vacuity imlies the other one, provided the elements of the latter are the contradictories of those of the former" (idem, p. 76, El-Rouayheb, p. 132).
- According to El-Rouayheb, al-Khūnajī had already expressed this law (El-Rouayheb, 2010) [presumably in *Kashf al-asrar*, since it is not found in *al-Jumal*].

15. '~~P ⊃ P' : the law of double negation, (idem, p. 72).
16. (P ∧ Q) → P plus '(P ∧ Q) → Q'

Plus some implications involving quantified propositions, for instance, the following:

17. Always if Some A is B, then Q → Always If every A is B then Q (El-Rouayheb, 132)
18. Always If P then Every A is B → Always If P then Some A is B (El-Rouayheb, 132)

In both cases, 'every A is B' should have an import in order for the formula to be valid.

Conclusion

- In what preceeds we have shown that the hypothetical logic has evolved in a significant way in the Arabic tradition.
- For while al-Fārābī presents just the Soïc indemonstrables and some of their variants, Avicenna builds several systems comprising different kinds of hypothetical moods and inferences.
- These systems had a great influence on his followers who introduced inside the same kind of frames some changes in the definitions and the inferences.
- With al-Khūnajī first and afterwards Ibn ʿArafa and al-Sanūsī, Avicenna's intensional definitions of the conditional and the disjunction are clarified and became more and more formal and extensional.
- This clarification and 'extensionalization' gave rise to some complex inferences that are not explicit in Avicenna's frame.
- Some laws of propositional logic are thus stated explicitly by these logicians: the De Morgan's laws, the law of double negation, the law of distributivity of disjunction over conjunction etc…plus implications with quantified propositions.
- There is thus a clear departure from intensionality. But this departure remains partial, since some definitions still have semantic features.

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Thank you!