Efforts to make mathematics infallible

Wilfrid Hodges

Queen Mary, University of London

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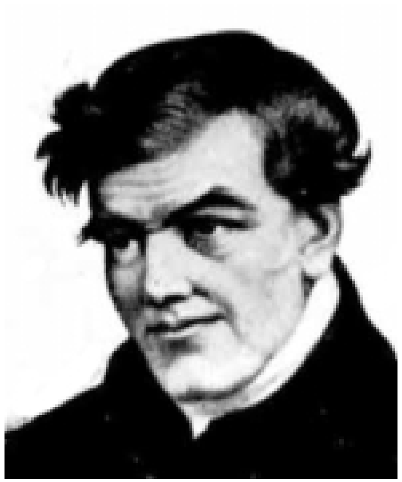
A cognitive commentary on the reorganisation of mathematics, 1850—1950

1. The reorganisation
2. The contribution of logicians, and their aims
3. Three semantic ingredients

1. The reorganisation as seen by mathematicians

- Symbol versus interpretation
- Classes of structure

Symbol versus interpretation



George Peacock
1791–1858

SYMBOL

Operations (increase)

Rules (permanent)

INTERPRETATION

Natural numbers

Integers

Reals

Complex numbers

Interpretation is something we do with our minds, not on the page.

David Hilbert (1899):

> [In geometry] the primitives can be thought (*gedacht*) in any way one likes. If I think (*denke*) of my ‘points’ as any system of things, for example love, law, chimney-sweeps . . .

Classes of structure

Evariste Galois (1832), aged 20, invents most of undergraduate group theory, decades before anybody else can understand it

The ‘objects’ are no longer numbers, but e.g.

- arbitrary elements of groups,
- arbitrary groups,

and since the 1920s

- relations and functions between groups,
- correspondences between group homomorphisms and continuous maps of topological spaces.

Need a background universe in which all this takes place.

(Bourbaki 1930s) The universe is that of set theory.

All mathematical definitions and assumptions reduce to set theory.

Earlier 'theories' (e.g. of groups) become definitions of classes of set-theoretic structure.

Proofs understood as informal versions of deductions in formal set theory.

Rival architectures (e.g. category theory, Lawvere 1970s) not radically different.

2. Aims of the logicians

Many logicians contributed to the reorganisation and put their own glosses on it.

Historically, logic claimed to teach us how to avoid errors of reasoning.
This was a constant theme in logicians' contributions to mathematics.

'Now it is the object of Method in Logic to determine in a precise manner and for our practical guidance what the formal conditions of validity of inference and generally of correctness in the operations of thought are.' (Boole 1857 unpub.)

'Any book whatever, even one full of blunders, may be made rigorous by leaving out what is false; what remains is the useful part of the book.' (Peano 1910)

'[Logical rules of inference are] infallible, i.e. always lead from true sentences to true sentences.' (Tarski 1936)

Main questions:

How did logicians aim to remove faulty reasoning from mathematics?

Did it work? If so, how?

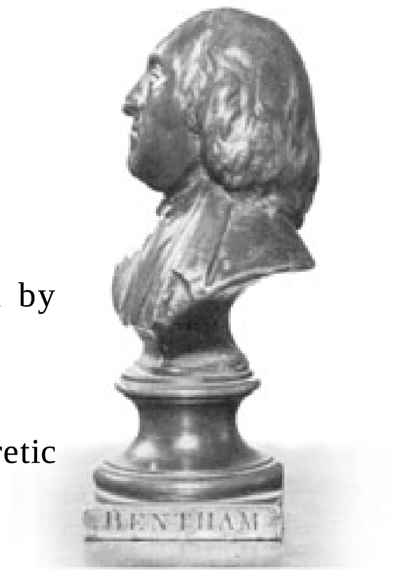
3. Three semantic ingredients

From Aristotle onwards, it was agreed that errors of reasoning arise from confusion of meanings.

‘Inference from two premises very often, if not always, depends on a concept being common to both of them. If a fallacy is to be avoided, not only must the sign for the concept be the same, it must also mean the same.’

(Frege 1898 unpub., but apart from the distinction between concept and meaning, it could have been almost any Aristotelian logician)

3(a). Use of count nouns



Jeremy Bentham (1811, reported in 1826 by his nephew George Bentham) proposed a method for avoiding ‘fallacies’:

- ‘The ... terms must ... be expressed by nouns-substantive’ which name aggregates of individuals;
- the verbs should express only set-theoretic relations between these aggregates.

Related example (not Bentham’s):

> My father-in-law’s birthday was the fourth of July.
> Americans celebrate the fourth of July.
> Therefore Americans celebrate my father-in-law’s birthday.

We escape the fallacy by asking:
What are the classes of entities
(i) that my father-in-law’s birthday is said to be one of,
(ii) that Americans are said to celebrate one of?

Around 1840 William Hamilton (of Edinburgh),
after reviewing Bentham’s book,
claimed the discovery as his own and added in lectures:

> We can reduce set-theoretic relations to symbols and calculate their consequences mechanically with the symbols. (My paraphrase)

Hamilton’s work is a direct antecedent of
the set-theoretic calculus of Boole and
the relation calculi of De Morgan and Peirce.

Problem: How to express transitive verb phrases (e.g. ‘is less than’) by count nouns?

Solution (Norbert Wiener 1914): Use set theory to define ordered pairs. Express ‘is less than’ by naming the class of ordered pairs (x, y) such that x is less than y.

From this point onwards, translation into count nouns coincides with the embedding of mathematics in set theory (noted above).

Not all logicians were convinced that the set-theoretic turn would improve reasoning.

‘The *must be* and the *cannot be* are the strong points of our mental constitutions. We know all about *can* and *cannot* from our cradles; we never feel the same assurance about *is* and *is not*.’ (De Morgan 1860)

Cf. Stenning and Lambalgen (2001) on abstract versus deontic tasks.

3(b). Removal of semantic notions

‘Interpretation’ in Peacock and Hilbert was something mathematicians do in their minds, not on the page.

From the 1890s, Hilbert and others began using re-interpretation as a mathematical method. Re-interpretation was a thing consenting mathematicians did in private, so there were no controls.

> ‘In any case, a more precise formulation seems necessary.’ (Frege, reply to Hilbert on geometric interpretations)

The solution came via the grammar of formal languages designed for set theory.

Tarski (1933) assigns to each grammatical expression ϕ a set $|\phi|$ (of sequences) as semantic value.

To each ‘fundamental operation’ (i.e. grammatical construction) Tarski assigns a set-theoretic operation on semantic values. E.g.

$$|\phi \text{ and } \psi| = |\phi| \cap |\psi|.$$

Outcome: a purely set-theoretic definition of ‘Assignment A of sets to primitive symbols in sentence ϕ makes ϕ true’.

The mind making the assignment becomes irrelevant.

Remark

This work of Tarski introduced **compositional semantics**.
A clear historical path leads from Tarski through Quine and Chomsky to the introduction of the term 'compositional' by Katz and Fodor.
The notion couldn't have been defined in any generality before the introduction of recursion on syntax (1921, Post and Wittgenstein).

Has Tarski removed minds from mathematics?

Clearly not, as long as mathematicians still *think*.

In fact Tarski has represented within set theory
the *concept* of interpreting symbols.
A mathematician using Tarski's formalism in the intended way *thinks about* interpretations instead of just making them.

Tarski does make possible the option of calculating in set theory purely as a formal system.
A mathematician doing this *thinks about* the formulas instead of just using them.

Corollary A description of the formal rules
followed by a reasoning mathematician
leaves wide open the question
what the mathematician is thinking.

NB David Marr (Vision 1982), describing the level of representation and algorithm, defines a representation as a formal system *together with a description of how it is applied.*

In some sense, modern mathematics hides the actual reasoning.

3(c). Inferences from not necessarily true premises

This was a practical necessity, widely regarded by logicians as dangerous for sound reasoning.

Causes:

- Reasoning from axioms, we don't have to interpret the axioms. So our premises aren't about anything.
- Very complicated arguments need to be broken down into pieces, perhaps handled by different mathematicians.

Jaśkowski and Gentzen (1930s) developed logical formalisms for the relation

ψ follows from the premises $\phi_1, \phi_2, \ldots$

independent of whether the premises are true or not.

Natural deduction and the sequent calculus are two such formalisms.

Why have some logicians regarded this as a threat to sound reasoning?

Frege's analysis:

After a step in reasoning, there has to be an answer to the question 'What have we established in this step?'

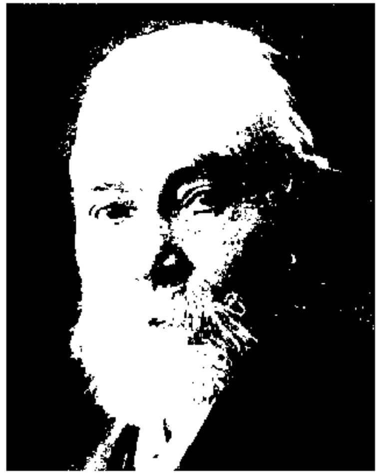
In reasoning 'ϕ, therefore ψ' it looks as if we established ψ. But we didn't; all we established was 'Under any interpretation, if ϕ then ψ'.

So strictly that is how we should state the conclusion.

Tarski's analysis shows how we can express 'Under any interpretation … ' precisely.

But this converts a single line of text into a whole page.

Frege himself noted that his preferred formulations might have 'eine ungeheuerliche Länge'.



Frege's stance resonates with Luria's subjects (1976):

E. In the Far North, where there is snow, all bears are white. Novaya Zemlya is in the Far North and there is always snow there. What color are the bears there?

A. We always speak only of what we see; we don't talk about what we haven't seen.

E. But what do my words imply?

A. Well, it's like this: our tsar isn't like yours, and yours isn't like ours. Your words can be answered only by someone who was there, and if a person wasn't there he can't say anything on the basis of your words.

What are mathematicians doing when they reason from unassumed premises?

Is this something one really does outside mathematics?

Before the modern mathematical examples, philosophers always thought it had something to do with counterfactual reasoning. (Some still do.)

A closing comment

The linguist As-Sīrāfī, debating with the logician Mattā ibn Yūnus at the court of Ibn al-Furāt in Baghdad, said (930):

> Your books are full of nonsense,
> because there is no way to create a new language
> within an already established language.

Mattā had claimed that the concepts of logic help us to avoid error. As-Sīrāfī answers that the concepts of logic are themselves defined within our established and supposedly fallible language.

Georg Kreisel comments that dirty dishes and dirty dishwater create clean dishes. What’s the trick?