How reasoning depends on representations

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Good thinking needs good representations.
(A fact noted by Archimedes in his *Sand-Reckoner*.)

- When the representations are public,
  we can measure their efficiency.
  I report some recent work relating representations in logic to different thinking strategies.

- When the representations are internal, experimental work is hard to tie up with introspective evidence. I imagine Kurt Gödel—logic's deepest introspecter—commenting on some recent cognitive science.

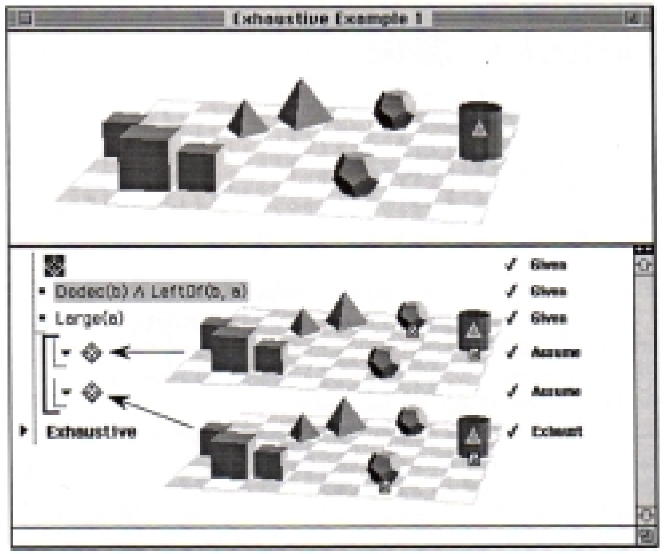
FIRST PART: PUBLIC REPRESENTATIONS

Hyperproof, by Jon Barwise and John Etchemendy

A proof calculus for first-order logic, using a computer.

Formulas can be represented by graphics using objects on a squared board.

Proofs can be carried out either with formulas or with the board.



Hyperproof can be used in two ways:

(1) in graphic mode, i.e. with the graphics enabled;
(2) in syntactic mode, i.e. with only formulas showing on the screen.

In syntactic mode, Hyperproof is a conventional natural deduction proof system.

Students using Hyperproof at Stanford were examined at the end of the course, using a computer in the exam.

K. Stenning, R. Cox and J. Oberlander (1995):

Students at Stanford University (all high achievers) were grouped semi-randomly into two groups, G and S.

Then the students took a one-quarter course in Hyperproof; students in group G received the graphic mode, S received the syntatic mode.

All the students were tested before and after the course with two types of question, all written in English without symbols:

- Blocks-world: problems about arrangements of pieces on a board.
- GRE analytical: problems for which a picture might be helpful.

(Also on GRE logical tests, but we ignore these here.)

Findings:

(a) The students in both group G and group S significantly improved their scores on the GRE analytical test.

(b) Students who started high on the GRE analytical test improved their scores on Blocks-world if in group G; if in group S their Blocks-world scores went down.

(c) Students who started low on the GRE analytical test improved their scores on Blocks-world if in group S; if in group G their Blocks-world scores went down.

(d) During the course exam,
students in group G who started low on the GRE analytical test moved into graphics mode and stayed there;
students in group G who started high on the analytical test moved often between graphics mode and formula mode.

This research has been partly replicated at Göteborg (Sweden),
and analysed further by Padraic Monaghan.

SECOND PART: MENTAL REPRESENTATIONS

Deduction, by P. N. Johnson-Laird and Ruth Byrne (1991)

Johnson-Laird and Byrne (JL-B) contrast

> using formal rules, which are sensitive only to the syntactic form of expressions

with

> a way to make deductions that depends, not on the form of expressions, but on their meanings.

They illustrate the latter as follows.

Problem: To make deductions from the following premises.

> Arthur is in Edinburgh or Betty is in Dundee, or both.
> Betty is not in Dundee.
> If Arthur is in Edinburgh, then Carol is in Glasgow.

For short,

> a or b, or both.
> Not b.
> If a then c.

(Their abbreviation, not mine.)

STEP ONE. List ‘the set of possibilities’ as follows:

	a	b	c
$i.$	T	T	T
$ii.$	T	T	F
$iii.$	T	F	T
$iv.$	T	F	F
$v.$	F	T	T
$vi.$	F	T	F
$vii.$	F	F	T
$viii.$	F	F	F

STEP TWO.

- 'a or b, or both' rules out lines vii and $viii$.
- 'Not b' rules out lines i, ii, v and vi.
- 'If a then c' eliminates iv.

STEP THREE. Only iii remains,
and since iii makes c true we can conclude c:
Carol is in Glasgow.

David Marr, *Vision* (1982), contrasts two levels of description of a cognitive system:

- The *computational theory* describes what problems are solved by the system, and the strategies involved.
- The *representation and algorithm* describes in formal terms how the system derives its output from its input.

Johnson-Laird and Byrne claim that their notion of 'model-theoretic reasoning' (illustrated above) lies at the algorithm level.

An algorithm in Marr's sense seems to be close to a *formal system* as discussed by Gödel.

A formal system has a precisely defined finite vocabulary, a recursive syntax, an explicit recursive set of axioms, and a set of explicit computational rules for deriving consequences from the axioms.

Thus Marr describes an algorithm for addition:

> . . . we might choose Arabic numerals for the representation, and for the algorithm we could follow the usual rules about adding the least significant digits first and "carrying" if the sum exceeds 9.

Gödel might comment that we can use Gödel numbering to define a system in which the operation 'add one' is definable, etc. etc.

Gödel:

> What is missing [from *Principia*] is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs.

Marr:

> . . . the important point is that if the notion of different types of understanding is taken very seriously, it allows the study of the information-processing basis of perception to be made *rigorous* [his italics].

Kurt Gödel

As JL-B imply, we can formulate
their deduction procedure above as an algorithm.
Hence we can formulate it as a formal system.

Gödel notes two ways of treating a formal system:

1. The symbols have meanings, and the system rules describe operations justified by these meanings.

2. The symbols of the system have no 'content', and we operate the system by using 'considerations about finite combinations of symbols'.

The JL-B description of their procedure leaves it open whether they are describing 1 or 2.
But since they label the procedure as 'model-theoretic', probably they mean to exclude 2.

Hence the symbols in their account have meanings, and different meanings imply different representations of the logical problem.

We examine some possible meanings.

We ask first what JL-B mean by a 'possibility'.

Answer One: A possibility is a statement.
E.g. line vii is the statement

> Arthur is not in Edinburgh, Betty is not in Dundee and Carol is in Glasgow.

Then Step Two contains the inference:

> Arthur is in Edinburgh or Betty is in Dundee, or both. Therefore it is not the case that: Arthur is not in Edinburgh, Betty is not in Dundee and Carol is in Glasgow.

Problem: We are not told how this inference is made.
If by another similar procedure, we have a regress.

Answer Two: A possibility is a way things might be.

Problem: This is a higher-level concept. To reason with it, we need further concepts, e.g. 'possible relative to given information'.

Gödel proved in 1936 that passing to higher-level concepts allows proofs to be speeded up, sometimes dramatically.

But to exploit this extra power would imply a boosting of the formal system.

Example (George Boolos)

$$\forall n\ F(n,1) = S(1).$$
$$\forall x\ F(1,S(x)) = S(S(F(1,x))).$$
$$\forall n \forall x\ F(S(n),S(x)) = F(n,F(S(n),x)).$$
$$D(1).$$
$$\forall x\ (D(x) \rightarrow D(S(x))).$$
$$\vdash\ D(F(S(S(S(S(1)))),S(S(S(S(1)))))).$$

Boolos gives a one-page proof in second order logic.
(One can make it a set-theoretic proof that uses induction.)

He also shows that
any proof in any standard proof calculus
would be of astronomical length.

Answer Three: A possibility is a function assigning one of True, False to each of the sentences a, b, c.

(This would be genuinely ‘model-theoretic’.)

Problem: What is meant by ‘ruling out’ or ‘eliminating’ a possibility?

Reasonable explanations fall back on Answer One.

Closing remarks

Much of Gödel’s introspection was about how we *know* mathematical facts. It’s implicit in his approach that our thinking gives us not just conclusions, but *evidence for* those conclusions.

I hope that cognitive scientists will allow this notion.

Example Frege claims that the axioms of his Begriffsschrift (concept-script) need no further justification once the concepts have been explained.

One axiom is

$$\vdash (c \to (b \to a)) \to ((c \to b) \to (c \to a)).$$

Don’t we in fact justify this, to ourselves and others, by a truth table calculation?
So the justification introduces other concepts, close to those we saw in JL-B.

There is surely a cognitive question here?

Hyperproof and its implications:

Jon Barwise and John Etchemendy, *Hyperproof*, CSLI, Stanford 1994.
K. Stenning, R. Cox and J. Oberlander, ‘Contrasting the cognitive effects of graphical and sentential logic teaching; reasoning, representation and individual differences’, *Language and Cognitive Processes* 10 (1995) 333–354.
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P. N. Johnson-Laird and Ruth Byrne:
Deduction, Lawrence Erlbaum Associates, Hove, 1991.

Formal systems and algorithms:
Kurt Gödel, 'Is mathematics syntax of language?', *Collected Works III, Unpublished Essays and Lectures*, ed. Solomon Feferman et al., Oxford University Press, New York 1995, pp. 334–362.
David Marr, *Vision*, W. H. Freeman, New York 1982.

Speed-up:
Kurt Gödel, 'Über die Länge von Beweisen', in *Collected Works I, Publications 1929–1936*, ed. Solomon Feferman et al., Oxford University Press, New York 1986, pp. 396–399.
George Boolos, 'A curious inference', *The Journal of Philosophical Logic* 16 (1987) 1–12; reprinted in George Boolos, *Logic, Logic, and Logic*, Harvard University Press, Cambridge Mass. 1998, pp. 376–382.