Indirect proofs and proofs from assumptions

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In his valuable book on mathematics and its philosophy in the seventeenth century [9], Paolo Mancosu includes a section '4.3. Proofs by contradiction from Kant to the present'. He comments (p. 105):

> . . . it is very difficult to ascertain in each single case what kinds of inferences the auithor wants to consider direct as opposed to apagogic. Indeed, even the meaning of proof by contradiction (apagogical proof, reductio ad absurdum, reductio ad impossibile, reductio ad incommodum, will be used as synonyms) is not quite clear.

This is certainly true. But it seems to me that one confusion between types of proof is due more to Mancosu himself than to his sources, and it leads him into mistakes when he interprets nineteenth and twentieth century authors.

1 Assumption proofs

By a *proof by assumption*, or for short an *assumption proof*, I mean the following. The proof proves a conclusion of the form 'If P then Q'. It proceeds by making a sequence of statements $S_1, \ldots, S_n$ (possibly with added commentary), where P is S_1, Q is S_n, and for each S_i with $i > 1$ there are earlier $S_j \ldots$ such that the conditional 'If $S_j \ldots$, then S_i' is already known.

The reason for calling these arguments 'proofs by assumption' is that they often begin with a phrase 'Assume' or 'Suppose' or 'Let', though one does find examples without this decoration. There are also quite a lot of examples where the assumptions consist of S_1 and S_2 together, so that the statement proved is 'If S_1 and S_2 then Q'. I ignore this refinement in what follows.

The statement being proved quite often takes one of the forms

If there is x such that $P(x)$ then Q.
If $P(x)$ then $Q(x)$.

In both cases it's common practice to start the assumption proof with a phrase like

Let t be such that $P(t)$.

In the second case the sentence is understood to have 'For all x' at the front. The same applies with more than one variable.

Here is a typical example from Euclid (Elements [2] Proposition i 15). The steps are a little jumbled with the commentary but the overall form is clear:

1. If two straight lines cut one another, they make the vertical angles equal to one another.
2. For let the straight lines AB, CD cut one another at the point E.
3. I say that the angle AEC is equal to the angle DEB.
4. For, since the straight line AE stands on the straight line CD, making the angles CEA, AED, the angles CEA, AED are equal to two right angles.
5. Again, since the straight line DE stands on the straight line AB, making the angles AED, DEB, the angles AED, DEB are equal to two right angles.
6. But the angles CEA, AED were also proved equal to two right angles.
7. Therefore the angles CEA, AED are equal to the angles AED, DEB.
8. Let the angle AED be subtracted from each; therefore the remaining angle CED is equal to the angle BED.
9. Similarly it can be proved that the angles CEB, DEA are also equal.

Sentence 1 is the statement being proved. Euclid reads it as if it had the form 'For all straight lines AB, CD, if they have a point of intersection E then . . . '. So in sentence 2 he says 'Let' and lays down five variables. (There is another 'Let' in sentence 8, but it plays a different role.)

For every assumption proof of a statement 'If P then Q', there is another proof that consists of the conditional statements 'If S_j . . . , then S_i'

that justify the steps in the assumption proof. We call this second proof the corresponding *conditional proof*; it proves the same conclusion as the assumption proof, but without assuming anything that hasn't been proved or granted.

The assumption proof and its corresponding conditional proof are obviously different pieces of text. But there is no evidence known to me that anybody before the twentieth century regarded them as stating different *proofs*. On the contrary there is evidence from all periods that they were seen as 'stylistic' variants (Frege's description).

To begin with Aristotle: his normal practice is to use the conditional style, as Łukasiewicz correctly says ([8] p. 1f). But in talking about proofs, he sometimes uses language that strongly suggests assumption proofs. Thus (Prior Analytics 61a19ff):

> A deduction through an impossibility is proved when the contradictory of the conclusion is put as a premise and one of the premises [of the deduction] is taken in addition . . .

But here is his example later in the same paragraph:

> For instance, if A belongs to every B and C is the middle, then if A is assumed to belong either not to every B, or to none, and to belong to every C (which was true), then it is necessary for C to belong either to no B, or not to every B. But this is impossible. Consequently, what was assumed is false.

'What was assumed' never appears as an assumption. Instead it is the antecedent of a conditional.

One could maintain either that Aristotle's assumption proof is in the object language and his conditional proof is a metalanguage description of it, or vice versa. But evidence for this is in short supply either way round.

Turning to Euclid, the following is typical (Elements [2] Proposition i 6):

1. If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

2. Let ABC be a triangle having the angle ABC equal to the angle ACB.

3. I say that the side AB is also equal to the side AC.

4. For, if AB is unequal to AC, one of them is greater.

5. Let AB be greater . . .

Here Euclid states in sentence 1 the conditional to be proved. In 2 he begins an assumption proof with ‘Let’. In sentence 4 he switches to conditional mode, and then in 5 he switches back to assumption mode with another ‘Let’. Clearly he is happy with both styles.

Proclus in his commentary on Euclid i [10] says (page 255):

> Every reduction to impossibility takes the contradictory of what it intends to prove and from this as a hypothesis proceeds until it encounters something admitted to be absurd and, by thus destroying its hypothesis, confirms the proposition it set out to establish.

This seems a clear enough description of a reductio ad absurdum argument where the premise ‘If not-P then contradiction’ is proved by an assumption argument. But here is Proclus’ own example of a reductio argument at the end of this same paragraph:

> For example, if in triangles that have equal angles the sides subtending the equal angles are not equal, the whole is equal to the part. But this is impossible; therefore in triangles that have two angles equal the sides that subtend these equal angles are themselves equal.

This is a straightforward modus tollens with no assumption argument visible at all. (Euclid’s original at Elements i Proposition 6 does incorporate an argument by assumption.)

The Port-Royal Logic of Arnauld and Nicole is more explicit. They have a section ‘On syllogisms whose conclusion is conditional’ ([1] iii.13). Their normal style for syllogisms is as follows, from this section:

> Every true friend must be willing to give up his life for his friends.
> But very few people are willing to give up their lives for their friends.
> So there are very few true friends.

They comment that in this layout the statements are ‘separated and laid out as at School’. But, they say, it is very common and very good (*très-commune et très-belle*) to express exactly the same reasoning as a conditional:

> If every true friend must be willing to give up his life for his friends, then there are very few true friends, because there are very few people willing to take it that far.

Moreover, they continue, this conditional form is appropriate when we grant the second premise of the syllogism but not the first. In this case we will be persuaded of the conditional conclusion but not of the conclusion of the original syllogism as stated.

Frege discusses proofs by assumption in several places, chiefly [3] p. 379f and [4] 156f. On this topic as on many others, he is very close to the tradition represented by the Port-Royal Logic, though he goes deeper. He makes a strong distinction between the logical question 'Are the logical steps in this argument sound?' and the epistemological question 'What do we know as a result of this argument that we might not have known before it?' His vocabulary is adjusted to this distinction. When all the steps in an argument from P to Q are sound, he calls the argument an *Ableitung* of Q from P. When an argument establishes that Q is true, using the fact that P is true, he says that the argument is a *Schluss* proving Q from P, and Q is the *Folgerung*. In these terms, a correctly constructed assumption proof that starts with P and finishes at Q is an Ableitung of Q from P, but not a Schluss proving Q from P. Rather it is a Schluss proving 'If P then Q', as is clear from the corresponding conditional form ([4] loc. cit.). If we want to make it into a Schluss proving Q from premise P, we need to tack onto it a proof of P ('Wir können sie als Bedingung nur los werden, wenn wir erkannt haben, dass sie erfüllt ist' [4]). So Frege's general view agrees exactly with that of Port-Royal.

Of course Frege differs from Port-Royal in having the experience of *Begriffsschrift* to support him; in *Begriffsschrift* only the conditional style is allowed. Where Port-Royal says that the conditional form is *très-belle*, Frege says rather that 'die Begriffsschrift durch ihre Übersichtlichkeit zur Wiedergabe des logischen Gewebes besser befähigt ist' [3]. Like Port-Royal, Frege regards the difference between the assumption form and the conditional form as largely stylistic ('stilistischen Gründen' [3]), but on the matter of style he points to an advantage of the assumption form: it avoids expressions of 'eine ungeheuerliche Länge' [3]. Evidently Frege is thinking of mathematics and the Port-Royal authors of rational conversation. Frege also comments that in the assumption style the propositions are ripped apart ([3] 'zerissen wird', recalling Port-Royal's 'separées et arrangées comme dans l'École').

Frege also makes two points not made by Port-Royal. The first is that assumption arguments often begin with statements containing free variables; Frege calls these statements *pseudopropositions*. In this case we can't even say that an Ableitung of Q from P establishes the truth of $P \rightarrow Q$, because $P \rightarrow Q$ lacks truth value. But, says Frege, it does establish the truth of

$$\forall xy \ldots \ (P \rightarrow Q)$$

where $x, y \ldots$ are the free variables in P and Q. (He gives an example in [3]. Actually he leaves out the quantifier. But since his point is that the conditional form is a genuine proposition, unlike its detached antecedent, he must be assuming that the variable is universally quantified. Leaving the quantifier implicit is in line with common mathematical practice, and with his use of 'latin' letters in *Begriffsschrift*.)

And second, Frege issues a warning about the language of 'assumptions'. It can lead to misunderstanding of the epistemological issue. Discussing an argument that in fact has no assumption proof in it, he says ([4] p. 158):

> So strictly speaking, we cannot say that consequences (*Folgerungen*) are being drawn from the false thought (not $AC > BC$). Therefore, we ought not really (*sollte eigentlich nicht*) to say 'suppose that not $AC > BC$', because this makes it look as though 'not $AC > BC$' was meant to serve as a premise for inference (*Schlüssen*), whereas it is only a condition.

In Frege's idiom this is a mild warning, not a prohibition. Frege comments at §12 of *Grundgesetze* that one 'ought strictly' (*muss eigentlich*) to fill in some steps that he leaves out in his proofs that follow.

Was there *really* anyone around who thought that you could prove a statement Q by making an arbitrary assumption and deriving Q from it? My own views on this have changed in Frege's favour since I wrote [5]. Several times a year now people send me from all over the globe their favourite refutations of Cantor, and it just is a fact that many of these refutations get in a muddle about assumptions in very much the way Frege warns against. These authors are certainly mathematical amateurs; some evidence suggests that most of them trained as philosophers, though there are also some computer scientists. One would hardly expect a professional to be so confused. But a hundred years ago standards in logic were much lower; Schoenflies and Jourdain (to name only two) were able to publish papers that wouldn't have got past first base with today's referees.

Let me elaborate a little. The number of people who simply assume P out of thin air, deduce Q from it and then immediately claim to have proved Q is vanishingly small. Frankly it would be evidence of insanity in anybody but prime ministers. But there is a larger number of people who do essentially this by moving everything to a metatheoretical level and getting themselves into a mental fog. Most of these people are probably too confused to benefit from Frege's careful admonishments.

2 Assumption proofs in the twentieth century

A Hilbert-style proof calculus is in a formal language. It has axioms and rules of inference. A proof consists of a sequence of formulas, where every formula is either an axiom, or is derivable by one of the rules of inference from some formula or formulas earlier in the sequence. Frege's *Begriffsschrift* is a Hilbert-style proof calculus.

As soon as one has such a calculus, a mathematical question arises. What is the relationship between the two following possibilities?

(a) If we add ϕ to the axioms of the calculus, ψ becomes derivable.

(b) The sentence $(\phi \to \psi)$ is derivable in the calculus.

One can think of (a) as describing a kind of assumption proof and (b) as a description of the corresponding conditional proof. But now the proofs are distinct formal objects, and it's a substantial question whether the existence of one implies the existence of the other. Curiously this question was asked only in 1921. Once asked it is not hard to answer. (b) implies (a) for any calculus with modus ponens as a rule of inference. For most Hilbert-style calculi, a proof by induction shows that (a) implies (b), provided that ϕ is a sentence with no free variables. This is Alfred Tarski's *deduction theorem* ([11] footnote p. 32).

A few years later, but still in Warsaw, Jan Łukasiewicz raised the question whether one could devise a logical calculus where the step from (a) to (b) is a rule of the system, not just a metatheorem. Thus Stanisław Jaśkowski [6]:

> In 1926 Professor J. Łukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method is that of an arbitrary supposition. The problem raised by Mr Łukasiewicz was to put those methods under the form of structural rules and to analyse their relation to the theory of deduction.

Jaśkowski showed how to do it, and thus was natural deduction born.

Łukasiewicz was rightly proud of having introduced into formal systems the rule taking an assumption proof directly to the proved conditional. It affected his judgement on Aristotle. He says ([8] pp. 2, 22):

> . . . no syllogism is formulated by Aristotle primarily as an inference, but they are all implications having the conjunction of the premisses as the antecedent and the conclusion as the consequent. . . . When we realize that the difference between a thesis and a rule of inference is from the standpoint of logic a fundamental one, we must agree that an exposition of Aristotelian logic which disregards it cannot be sound.

I think one has to suspect that Łukasiewicz is trying to apply to Aristotle distinctions which only became coherent when formal proof calculi were introduced.

Round about the middle of the twentieth century, uninterpreted first-order languages became the language of preference for formal theories. Suppose that by any proof calculus one derives a sentence ψ of such a language from another sentence ϕ. What does this prove? Applying Frege's recipe from [3] and following through the definitions, we see that ϕ and ψ are pseudopropositions with free variables for the features of a structure. So, universally quantifying over structures, a derivation of ψ from ϕ establishes that for every structure in which ϕ is true, ψ is true too; in short, every model of ϕ is a model of ψ. This is how model theorists usually understand such a derivation. But we also have the option of saying that the derivation establishes that ψ is formally derivable from ϕ. There need be no quarrel between these two readings, because the completeness theorem for first-order logic makes them equivalent.

3 Indirect proofs

Another kind of proof is *indirect proof*. Here, as Mancosu says, the boundaries are vague. But Kant [7] describes two forms that he counts as indirect. Thus (B819):

> ... darf man nur unter den aus dem Gegenteil derselben fliessenden Folgen eine einzige falsch finden, so ist dieses Gegenteil auch falsch, mithin die Erkenntnis, welche man zu beweisen hatte, wahr.

This seems to be a description of the argument form

> Given 'If not-P then Q' and not-Q, infer P.

(Kant says, not quite accurately, that this is modus tollens.) A little later Kant refers to a form of argument that he says is inadmissible in transcendental argument; in context he seems to be saying that it is admissible in mathematics. (B820)

> ... seine Behauptungen dadurch zu rechtfertigen, dass man das Gegenteil widerlegt.

This could be the argument form

> Given 'If not-P then contradiction', infer P.

This is essentially the rule that removes double negation.

Euclid's proof of Proposition i.19 in his Elements [2] is often quoted as an example of an indirect proof.

(a) In any triangle the greater angle is subtended by the greater side.

(b) Let ABC be a triangle having the angle ABC greater than the angle BCA;

(c) I say that the side AC is also greater than the side AB.

(d) For, if not, AC is either equal to AB or less.

(e) Now AC is not equal to AB;

(f) for then the angle ABC would also have been equal to the angle ACB;

(g) but it is not.

(h) therefore AC is not equal to AB.

(i) Neither is AC less than AB,

(j) for then the angle ABC would also have been less than the angle ACB;

(k) but it is not;

(ℓ) therefore AC is not less than AB.

(m) And it was proved that it is not equal either.

(n) Therefore AC is greater than AB.

There are two indirect arguments here. The inference from (f) and (g) to (e) is a modus tollens, and so is that from (j) and (k) to (i). As it happens, this argument is also an assumption argument in view of (b); but this feature is completely independent of the indirect subarguments.

There is no inherent connection between assumption proofs and indirect proofs. The examples above illustrate this. But since indirect proofs often have premises that are conditionals, an author might sometimes use an assumption proof in order to prove one of these conditionals. Not all authors do; the previous section contains examples of indirect proofs in Aristotle and Proclus where the proof of the conditional premise is written in conditional style. Another example is Frege's presentation of Euclid's proof of Proposition i.19 in [4]. Immediately after giving this proof, Frege explicitly warns against describing this proof as an assumption proof; we quoted this earlier.

Mancosu's book records a number of writers in the philosophy of mathematics who avoided indirect proofs for aristotelian reasons. Namely, an indirect proof shows that its conclusion must be true, but it doesn't show the reason why. These objections are to the indirect proof itself, not to the proof of the conditional premise of the indirect proof. I don't know of any writer who raised any objection to the propriety of using either the assumption or the conditional style for that part of the argument.

Mancosu includes Frege among those writers who had 'problems' with indirect proofs. This is puzzling. On the one occasion (as far as I know) when he does discuss indirect proofs ([4] p. 157f), Frege's conclusion is that the difference between direct and indirect proofs is no big deal: 'In Wahrheit ist der Unterschied zwischen einem direkten und einem indirekten Beweise gar nicht erheblich'. Curiously Mancosu quotes this, though he still claims that Frege shared Bolzano's project of eliminating indirect proofs ([9] p. 104). In fact all the standard forms of indirect proof are valid inferences in *Begriffsschrift*, and for Frege this is the highest guarantee of logical virtue.

4 Mancosu's commentary

Mancosu's definition of proofs by contradiction seems to require that their conditional premise is proved by an assumption proof:

> Minimally, [proof by contradiction] means a proof that starts from assuming as a premiss the negation of the proposition to be proved. From this premiss we then derive a falsity or, equivalently, a contradiction. We are then allowed to infer the proposition that had to be proved.

Also on page 117 he invites us to

> define an indirect proof as one that assumes a false formula, for example the negation of what we want to prove . . .

These definitions are probably harmless in talking about writers who don't regard assumption proofs and conditional proofs as different proofs. But they certainly don't transfer to the twentieth-century situation. They may also create a false impression that writers who had problems with indirect proofs also had problems with proof by assumption.

They also leave us in the dark about what the various authors mentioned by Mancosu without quotations (e.g. Lotze, Ueberweg) had in mind by indirect proofs. The proof from Crusius suggests that he goes with Kant and doesn't have assumption proofs in mind. Hessenberg (Mancosu's note 65 on p. 234) certainly talks about assumptions, but that's in 1912.

On page 105 Mancosu refers to the 'apparently queer pronouncements of Bolzano and Frege about proofs by contradiction, and ultimately reasoning under hypothesis'. I have no idea which of Frege's quoted pronouncements Mancosu finds 'queer'. Since Mancosu lumps together proofs by contradiction and reasoning under hypothesis, which are totally different matters, it's unclear which of these two topics he finds Frege 'queer' on. To my eye the passages of Frege that Mancosu quotes all look pretty straight. I recall that Frege says he has no problems with proofs by contradiction, and indeed with his view of logic he has no cause to have problems with them; so on this topic he is on the opposite side to Bolzano. I recall that on reasoning under hypothesis his views are close to those of Port-Royal; are the views of Port-Royal 'queer'?

On page 109 Mancosu ascribes to Frege

> the thesis that an inference must proceed by appealing to true premises; a statement that, prima facie, seems to exclude outright proof by contradiction from the realm of inferences.

There are two confusions here. The first is to think that inference from untrue premises has anything to do with proof by contradiction. I recall that in Frege's own example of an indirect proof in [4] there are no unproved premises, and that Frege himself points this out. His example uses modus tollens, but the same would apply with a full-blooded proof by contradiction.

The second confusion is to think that Frege rejected inference from other than true premises. This is to confuse the logical with the epistemological question that Frege struggled so hard to keep separate in his readers' minds. A logical derivation can be from a proposition that is false, or even a pseudoproposition. But a proof genuinely establishing that something is the case can't rest on assumptions that might be false.

Mancosu finishes this section of his book by noting that Gentzen showed how to reduce proofs in the natural deduction calculus NK to proofs in the calculus LK. He suggests that this reduction answers the question of 'reducing indirect proofs to direct proofs', and he says that Gentzen's reduction 'vindicates the reductions of Bolzano and Frege'.

The only reductions ascribed to Bolzano and Frege in Mancosu's text are the observations that one can convert a certain indirect proof of Euclid's Proposition i.19 to a direct proof. (This was an old observation; Proclus had already made it, though by a different reduction.) But reduction of indirect to direct proofs has nothing whatever to do with the relation between assumption proofs and conditional proofs. There are natural deduction calculi that allow indirect proofs and natural deduction calculi that reject them; they reduce respectively to Hilbert-style calculi that allow indirect proofs and Hilbert-style calculi that reject them.

Also Mancosu has chosen the wrong reduction. There are reductions of natural deduction calculi to Hilbert-style calculi, but the reduction to LK is not one of them. In fact LK is a sequent calculus; in this calculus every single line of a proof can be regarded as an assumption proof.

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