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Model Theory, Tarski and Decidable Theories

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The year is 1928.
We imagine ourselves in Alfred Tarski's seminar in Warsaw.

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Our problem: To analyse the additive group $\mathbb{Z}$ of integers.

We will use a method of research called
elimination of quantifiers,
developed since 1915 by Löwenheim, Skolem, Bernays, Langford and others.

Note: At this date model theory has not yet been invented.
Anatoliĭ Mal'tsev will invent it in 1940,
and Tarski will name it in 1954.

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Step One: Define the *domain*,
i.e. the set of individuals that formthe subject matter.
Our domain is the set of integers.

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Step Two: Determine the relevant primitive concepts. These are

1. The concept ‘integer’,
   expressed by a relation symbol $Z(x)$.
2. The concepts ‘zero’, ‘one’,
   expressed by individual constant symbols 0, 1.
3. The concepts ‘plus’ and ‘minus’,
   expressed by a 2-ary infixed function symbol $+$
   and a 1-ary function symbol $-$.

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The *language* of $\mathbb{Z}$, $L(\mathbb{Z})$ or L for short, consists of terms and formulas.

The set of *terms* of L is the smallest set τ of strings of symbols that contains

1. 0, 1;
2. each variable x_i (i a natural number);
3. $(s+t)$ where s and t are in τ;
4. $(-s)$ where s is in τ.

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The set of *formulas* of L is the smallest set Φ of strings of symbols that contains

1. $(s=t)$ where s and t are terms;
2. $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ where ϕ and ψ are in Φ;
3. $\forall x((\neg Z(x)) \vee \phi)$ and $\exists x(Z(x) \wedge \phi)$
   where x is any variable and ϕ is in Φ.

Note: In about 1950 Tarski dropped the use of $Z(x)$ and instead read $\forall x$ and $\exists x$ as ranging over all the individuals of $\mathbb{Z}$. We will do the same; this simplifies notation.

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Meanings of the logical symbols:

$=$	equals
$\neg$	not
$\wedge$	and
$\vee$	or (including ‘and’)
$\forall$	for all
$\exists$	there is

With these meanings we can read any formula as expressing something about $\mathbb{Z}$.

The formulas that say something true or false about $\mathbb{Z}$ are called *sentences*.

In fact sentences are those formulas in which every occurrence of a variable x is bound by a quantifier $\forall x$ or $\exists x$.

If a formula ϕ has some variables with free occurrences,
say $x_1, \ldots , x_n$,
then by assigning integers $a(x_1), \ldots , a(x_n)$ to these variables
we again make ϕ into a true or false statement about $\mathbb{Z}$.
The assignment a *satisfies* ϕ if this statement is true.

Main aim: We will assign to each formula ϕ
a *value* $|\phi|$ which will tell us

- (if ϕ is a sentence) whether ϕ is true or false;
- (if ϕ is not a sentence) which assignments satisfy ϕ.

The value $|\phi|$ will eventually be defined by recursion,
but we don't yet know what to recurse on.

Step Three. We choose a set of *basic formulas*.
These are formulas for which we can get the information about truth or satisfaction very easily.

Our intention is to choose each $|\phi|$ so that

- $|\phi|$ is a formula that is *equivalent to* ϕ *in* $\mathbb{Z}$,
  i.e. satisfied by the same assignments;
- $|\phi|$ is a boolean combination (i.e. by $\neg$, $\wedge$, $\vee$) of
  basic formulas;
- when ϕ is basic then $|\phi| = \phi$.

(3.1) First suggestion for basic formulas:
the *atomic formulas*, i.e. formulas of the form $(s = t)$.

If integers are assigned to all variables in s or t,
then we can easily calculate the integer values of s and t,
and so check whether the statement $(s = t)$ is true.

Step Four: We choose a set of *axioms*.
These are sentences that satisfy three requirements:

- 'Their truth should appear evident to us.' (Tarski, Ch. 6 'On the deductive method', in 'Introduction to Logic and to the Methodology of the Deductive Sciences', 1936.)
- They should be adequate to prove all true basic sentences and the negations of all false basic sentences.
- They should be adequate to prove all the sentences

$$\forall x_1 \ldots \forall x_n((\phi \wedge |\phi|) \vee ((\neg\phi) \wedge (\neg|\phi|)).$$

(4.1) First attempt at axioms:

$$\begin{array}{ll} \zeta_{\text{ass}}: & \forall x_0 \forall x_1 \forall x_2(((x_0 + x_1) + x_2) = (x_0 + (x_1 + x_2))). \\ \zeta_{\text{com}}: & \forall x_0 \forall x_1((x_0 + x_1) = (x_1 + x_0)). \\ \zeta_{\text{zero}}: & \forall x_0((x_0 + 0) = x_0). \\ \zeta_{\text{inv}}: & \forall x_0((x_0 + (-x_0)) = 0). \end{array}$$

These axioms are all true,
and they allow us to use abelian group notation.

Also they prove all true basic sentences.

Do they prove all negations of false basic sentences?
NO. They don't prove $0 \neq 1$, $0 \neq 2$ etc.

So we need more axioms.

(4.2) Second attempt at axioms:
The previous axioms plus

$$\begin{array}{ll} \zeta_{\text{one}}: & \neg(0 = 1). \\ \theta_n: & \forall x_0 \ (nx_0 = 0 \rightarrow x_0 = 0) \ \ (n > 1) \end{array}$$

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Step Five: We try to define $|\phi|$ for all formulas ϕ.

If ϕ is basic, or a boolean combination of formulas ψ for which $|\psi|$ is defined, then we know what to do.
For example we put

$$|(\neg\psi)| = (\neg|\psi|).$$

The significant case is where ϕ begins with a quantifier (hence the name *quantifier elimination*).

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Now assuming ψ is a boolean combination of basic formulas, we can put the boolean combination into disjunctive normal form, say $(\psi_1 \vee \ldots \vee \psi_m)$
where each ψ_i is a conjunction $(\chi_1 \wedge \ldots \wedge \chi_k)$, and each χ_j is either a basic formula or a negation of a basic formula.

Since $\exists x(\psi_1 \vee \ldots \vee \psi_m)$ is equivalent to $(\exists x\psi_1 \vee \ldots \vee \exists x\psi_m)$, we can assume ϕ is

$$\exists x(\chi_1 \wedge \ldots \wedge \chi_k)$$

with the χ_j as before. Such a ϕ is called a *p.p.* formula.

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Example: ϕ is

$$\exists x_0\ (5x_0 + 3x_1 + 2 = 0 \wedge 3x_0 \neq 2x_2).$$

Since the axioms say we have a torsion-free abelian group, we can multiply through and rearrange:

$$\exists x_0\ (-3x_1 - 2 = 15x_0 \wedge 10x_2 \neq 15x_0).$$

This is equivalent to

$$(\exists x_0\ (-3x_1 - 2 = 15x_0) \wedge 10x_2 \neq -3x_1 - 2)$$

so we need only define $|\exists x_0(-3x_1 - 2 = 15x_0)|$. Can we?

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Here follow three days of attempts to find
a boolean combination of basic formulas equivalent to

$$\exists x_0(y = 15x_0).$$

On day four we give up
and admit we need to expand the class of basic formulas.

(3.2) Second suggestion for basic formulas:

- The atomic formulas, as before.
- All formulas

  $$\exists x(t = p^n x)$$

  where t is a term, p a prime and n a positive integer.
  We write this formula as

  $$(p^n|t)$$

We have to go back and check earlier results.

Do our axioms still prove all true basic sentences? YES.

Do our axioms prove the negations of all false basic sentences? NO.
They don't prove $(2 \nmid 1)$, i.e. $\neg\exists x(1 = 2x)$.

So we must expand the axioms again.

(4.3) Third attempt at axioms:
The previous axioms plus

$$\eta_p : \quad (p \nmid 1) \;\; (p \text{ any prime})$$

The new axiom set does prove all negations of false basic sentences.

Unfortunately we now have new p.p. formulas.

Example:

$$\exists x_0(5x_3 = 2x_0 \wedge (2^2|x_0) \wedge (2^3 \nmid x_0)).$$

As before, we can multiply through and rearrange:

$$(\exists x_0(5x_3 = 2x_0) \wedge (2^3|5x_3) \wedge (2^4 \nmid 5x_3)).$$

$$((2|5x_3) \wedge (2^3|5x_3) \wedge (2^4 \nmid 5x_3)).$$

Success!

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Example:

$$\exists x_0(4x_0 \neq x_1 \wedge (7|x_0+3) \wedge (13 \not| 5x_0 - x_2)).$$

Here we can't use an equation to remove x_0 from all other conjuncts.
Instead we show that if $x_0 = k$ solves $(7|x_0+3) \wedge (13 \not| 5x_0 - x_2))$ (for a given value of x_2), then so does $x_0 = k + (7 \times 13)m$ for any m.

So the inequality is irrelevant and can be ignored.

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Example:

$$\exists x_0\ ((2 \not| x_0) \wedge (2 \not| x_0 + 1)).$$

We know this is false, but can we prove that from the axioms? NO.

(4.4) Fourth attempt at axioms:
The previous axioms plus

$$\kappa_p : \quad \forall x_0((p|x_0) \vee \ldots \vee (p|x_0 + p - 1)) \quad (p \text{ prime})$$

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Step Six: Check that this is enough.
With this set of axioms and this set of basic formulas, we can define $|\phi|$ as required, for all formulas.
Then write it all out and submit it for publication.

The published version will need a definition of $|\phi|$ by recursion on some kind of rank.

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We can define a suitable rank by putting:

- basic formulas have rank 0;
- a boolean combination of formulas with at least one of rank $\geqslant r$ has rank $\geqslant r$;
- if ϕ has rank $\geqslant r$ then $\exists x\phi$ has rank $\geqslant r+1$.

The rank of a formula is the minimum possible by this definition.

Tarski's student Mojżesz Presburger did this work in 1928 and published in 1930.

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Tarski and students applied quantifier elimination to:

- The field of real numbers
- Algebraically closed fields of a given characteristic
- Boolean algebras
- The natural numbers with $+$
- Abelian groups.

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Application 1: Given any sentence ϕ of L,
we can compute $|\phi|$ and then calculate whether $|\phi|$ is true.
This gives a *decision procedure* for first-order statements about the additive group of integers.
In 1957 Martin Davis implemented it on the Johnniac computer.
In 1973 Michael Fischer and Michael Rabin showed that any algorithm solving this problem must have super-exponential time complexity.

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Application 2: The equivalence of ϕ and $|\phi|$ is provable from the axioms.
If ϕ is true then $|\phi|$ is also provable from the axioms, and so ϕ itself is provable from the axioms.

Hence the axioms allow us to prove every true statement of L.

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Application 3: Suppose A is another structure for which the language L makes sense,
e.g. another group.
We say A is *elementarily equivalent* to $\mathbb{Z}$, $A \equiv \mathbb{Z}$,
if the same sentences of L are true in both structures.

We have shown that $A \equiv \mathbb{Z}$ if and only if all the axioms are true in A.
For example $\mathbb{Z} \equiv \mathbb{Z} \bigoplus \mathbb{Q}$.

In 1949 Tarski published several results of quantifier elimination, for example for algebraically closed fields.
In his PhD thesis (1949) Abraham Robinson independently found much more algebraic ways of getting the same results,
using the compactness theorem for first-order logic and homomorphisms between structures.
So Robinson’s methods were *model-theoretic*,
as opposed to Tarski’s syntactic approach.
Later model-theoretic approaches (e.g. back-and-forth) eliminated the compactness theorem.

Application 4: Definition of truth.
In 1930 Tarski saw that we get a definition of ‘true sentence of a formalised language’ if we can define $|\phi|$ suitably, by recursion on the complexity of ϕ.
He had the idea of taking $|\phi|$ to be
the set of assignments that satisfy $|\phi|$.
Later semanticists generalised this:
the *meaning* $|\phi|$ of a complex expression ϕ is defined in terms of $|\psi|$ for the immediate constituents ψ of ϕ.
A recursive definition of this kind is called *compositional*.

Sadly, Tarski didn’t always appreciate the advances built on his work.

In 1972 he told Barbara Partee that his truth definition was not compositional. (Of course it is.)

In 1977 he said that the method of quantifier elimination gives

> a more direct and clearer insight than the modern more powerful methods.

References

- Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry, University of California Press, Berkeley 1951.*
- Wilfrid Hodges, *A Shorter Model Theory*, Cambridge University Press 1997, §2.7.

These notes are at `www.maths.qmul.ac.uk@~wilfrid@mumbaitutorial.pdf`.