In memory of Maria Panteki

How Boole broke through the top syntactic level

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`wilfridhodges.co.uk/history14.pdf`


Happy New Year
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The mathematical historian

- discovers and edits documents;
- traces influences;
- assesses writers in the light of their own context.

The history-minded mathematician

- uses the documents provided;
- traces discoveries and mistakes;
- assesses writers in the light of the 'mathematical facts'.

During the period 1830–1930 (roughly),
logic changed from Traditional to Modern.
What did this change consist in?
What were the real differences?

Hazard 1: As after any fight to the death,
the victor's propaganda becomes the historical record —
unless and until it is challenged.

Hazard 2: Łukasiewicz's paradigm for history of logic
has emphasised formal systems at the expense of
usage, purpose, justifications etc.

Top-level processing

A feature of traditional logic:

In formal inferences, only the top syntactic level of the premises is unpacked.

Every A is a B. Every A is a C. Therefore some B is a C.

Details in Wilfrid Hodges,
'Traditional logic, modern logic and natural language',
to appear in *Journal of Philosophical Logic*.

The moment of breakthrough

George Boole *Mathematical Analysis of Logic* (1847) p. 67:

> Now the most general transformation of [the equation $a_1t_1 + a_2t_2 \ldots + a_rt_r = 0$] is
>
> $$\psi(a_1t_1 + a_2t_2 \ldots + a_rt_r) = \psi(0)$$
>
> provided that we attribute to ψ a perfectly arbitrary character, allowing it even to involve new elective symbols, having *any proposed relation* to the original ones.

We call this Boole's Rule.

For example $\psi(0) = fghjk(0)$, or

$$\psi(0) = f\Big(g\big(h(j(k(0)))\big)\Big)$$

Why is this breakthrough important?

A modern form of Boole's Rule is the rule

$$s = t \;\vdash\; \phi(s/x) \rightarrow \phi(t/x).$$

where no quantifier in ϕ captures the variables in s or t.

We can get a completeness theorem for first-order logic with this rule restricted to atomic formulas ϕ.

But ...

To make substitutions below top level,
we have to be able to break down formulas,
in particular removing quantifiers.

Traditional logic was very bad at handling formulas
with unquantified variables.
(Leibniz did it, but even he didn't write down the rules.)

As far as we know, the rules for quantifiers need
substitutions at arbitrary depth.
For example Shoenfield *Mathematical Logic* (1967) p. 21
has axioms

$$\phi(s/x) \to \exists x \phi$$

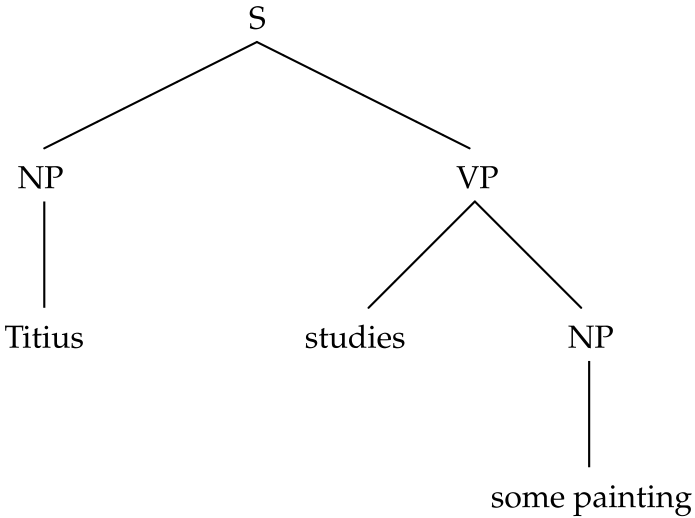
Boole was not the first to challenge Top-Level Processing. From the 14th century, logicians tried to extend syllogistic rules to parts of sentences. Example (Leibniz):

> Painting is an art. Therefore a person who studies painting studies an art.

By quantifier rules that Leibniz understood,
it suffices to show:

> All painting is an art. Titius studies some painting. Therefore Titius studies some art.

Snag: 'some painting' is not nominative
(because not at top syntactic level).


S
NP
VP
Titius
studies
NP
some painting

Leibniz's solution:

Paraphrase!

All painting is an art.
Some painting is a thing that Titius studies.
Therefore some art is a thing that Titius studies.

The step of paraphrasing rests not on logic but on linguistic intuition.

> It should also be realized that there are *valid non-syllogistic inferences* which cannot be rigorously demonstrated in any syllogism unless the terms are changed a little, and this altering of the terms is the non-syllogistic inference. There are several of these, including arguments from the nominative to the oblique … (*New Essays*)

Frege knew his Leibniz and condemned Leibniz's 'non-syllogistic' steps:

> [Our doubt about the analytic character of arithmetic] can only be canceled by means of a gapless chain of deductions, so that no step could appear in it that is not in accordance with one of a few inference principles that are recognized as purely logical. (*Grundlagen* 1884)

Frege's remedy was to allow substitutions *at any level* in a formula.

Boole shows no knowledge of any of this.

The syllogistic principle that the traditional logicians tried to generalise to deeper levels was:

> Given $\alpha \to \beta$ and $\phi(\alpha)$,
> if α is positive in $\phi(\alpha)$,
> then infer $\phi(\beta)$.

Leibniz and Frege both understood this.
Buridan (14th c) and De Morgan (19th c) didn't,
and made a pig's ear of the question.

Frege (and Shoenfield etc.) sidestepped the problems by using substitutions that don't depend on positivity.

Boole's Rule doesn't depend on positivity either.
Did he realise this, or why it mattered?

Boole *Mathematical Analysis of Logic* p. 3:

> They who are acquainted with the present state of the theory of Symbolical Algebra, are aware,
> that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible …

The reference is to Peacock's Symbolical Algebra 1833.
In Symbolical Algebra we manipulate symbols according to formal rules.
The results are true whenever we 'interpret' the symbols in ways that respect the rules.
Arithmetic 'suggests' rules that we should use.

Boole may have learned Peacock's ideas through Duncan Gregory.
But certainly Boole understood them better than either Peacock or Gregory.
For example 'system of interpretation' — Peacock and Gregory interpreted symbols one at a time.

Maria Panteki (letter 1999):

> Since you mention Boole, I found not a single reference of his to Peacock, and I was greatly surprised. There was definitely a line of influence from P's symbolic algebra to B's algebraic logic, but as noted in my paper this line concerned mainly the elaboration of P's ideas by D. F. Gregory. … Of course you have a specific prism to see their writings, that of model theory, a modern approach, whereas my own tends to be deeply historical, checking rather the background of these notions than their fruit.

Boole probably accepted his Rule because it worked in algebra and analysis.

> [In logic] there is even a remarkable exemplification, in its general theorems, of that species of excellence which consists in freedom from exception.
> And this is observed where, in the corresponding cases of the received mathematics, such a character is by no means apparent. (*MAL* p. 7)

!!

It's difficult to find in Boole even a definition of equations or an explanation of their properties.

> The general equation
>
> $$x = y$$
>
> implies that the classes X and Y are equivalent, member for member; that every individual belonging to the one, belongs to the other also. Multiply the equation by x, and we have
>
> $$x^2 = xy;$$
>
> $$\therefore x = xy$$
>
> (*MAL* p. 24)

Peacock didn't know what he meant by an equation.
Treatise on Algebra I (1842) p. 4:

> $=$ [denotes] equality, or the result of any operation or operations. …
> The sum of 271, 164, and 1023, or the result of the addition of these numbers to each other,
> is equal to 1458.

(Peacock's fatal tendency to try to have it both ways!)

Note also Peacock, *Treatise on Algebra I* p. 198:

Given

$$a_1 A_1 = \alpha_1 A_2, \;\; a_2 A_2 = \alpha_2 A_3, \;\; \ldots, \;\; a_{n-1} A_{n-1} = \alpha_n A_n$$

find the value x of $a_n A_1 / A_n$.
The solution is

$$x = \frac{\alpha_1 \alpha_2 \ldots \alpha_n}{a_1 a_2 \ldots a_{n-1}}$$

Peacock's proof removes $=$ altogether and uses the theory of proportions.

Conclusion

The person who first takes an important step
doesn't always see how to use it
or how to justify it.

But of course we knew this.