Oleg Belegradek and various groups and relations

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1. Photograph of a key moment

(sadly out of focus)

This was from the 6th Easter Conference on Model Theory, Wendisch-Rietz, April 1988.

Participants:

Fred Appenzeller, Andreas Baudisch, Oleg V. Belegradek, Christine Charretton, Bernd Dahn, Jörg Flum, Peter Göring, Siegfried Gottwald, Petr Hájek, Lutz Heindorf, Eberhard Herrmann, Wilfrid Hodges, Richard Kaye, Jiři Krajicek, Wolfgang Lenski, Christian Michaux, Daniele Mundici, Roman Murawski, Ludomir Newelski, Leszek Pacholski, Anand Pillay, Bruno Poizat, Jana Ryšlinková, Peter Tuschik, Martin Weese, Graham Weetman, Helmut Wolter, Martin Ziegler, Boris Zilber.

Oleg Belegradek contributed a paper

> ‘Some model theory of locally free algebras’.

It contains a theorem (joint with his student ‘V. A. Tolstych’) on the stability of completions of the theory of locally free algebras,
and a counterexample (joint with B. I. Zil’ber) to a conjecture of my ex-student Pillay.

2. Oleg's early breakthrough

In 1975 Oleg presented his Candidate thesis, written under Mikhail Taitslin. I assume its contents are represented by three papers, all written in Russian:

- 'On algebraically closed groups', *Algebra i Logika* 1974.
- 'On definability in algebraically closed groups', *Mat. Zametki* 1974.
- 'Elementary properties of algebraically closed groups', *Fundamenta Math.* 1978.

(A nontrivial group G is algebraically closed iff it is existentially closed, iff whenever H is a group $\supseteq G$ and ϕ is an $\exists$ first-order sentence true in H, with parameters from G, then ϕ is already true in G.)

His central result, proved in the 1978 paper using lemmas from the earlier papers, was:

> There are 2^ω algebraically closed groups,
> no two of which have the same $\forall\exists\forall$ first-order theory.

The proof is by constructing continuum many a.c. groups, no two of which contain copies of just the same finitely generated groups with r.e. word problem.

This answered several questions posed by Angus Macintyre in his paper of 1972 on algebraically closed groups.

Oleg's answer to Macintyre's questions was recovered independently by Martin Ziegler in his 1976 Habilitationsschrift. Ziegler only became aware of Oleg's 1978 paper late in the writing of his report in the 1980 Proceedings of the Oxford Conference on Word Problems (1976); he added it at the end of his references without comment in his text.

Ziegler's paper is probably where I first met Belegradek's name. In the West we had known of Boris Zilber (spelt Zilberg) from a footnote added in proof to a 1977 paper of Shelah. This footnote caused me to learn Russian.

Neither Oleg nor Boris was in the Russian delegation to the Word Problems conference.

3. Digression on some later developments

Ziegler derived various results on a.c. groups by using games based on Abraham Robinson's finite forcing.

In 1983 I lectured on this approach at the British Mathematical Colloquium in Aberdeen.
Graham Higman in the audience became interested.
He gave a course of lectures in Oxford, deriving Oleg's results and some new ones by use of games.

It was reported that Higman dedicated these lectures to 'The onlie begetter of these insuing results, Mr. W. H.'. (But rather than wishing me 'all Happinesse' he attributed a result of mine to Shelah. Never mind.)

Elizabeth Scott wrote up these lectures, and Higman and she published them together in *Existentially Closed Groups* 1988.

Meanwhile I published my own account in *Building Models by Games* 1985.

As a kind of dual to Oleg's main result (but shallower), I showed:

> There is a family of continuum many elementarily equivalent a.c. groups such that if a finitely generated group is embeddable in more than one of them then it has solvable word problem.

Ken Hickin saw this and gave a simpler proof without the elementary equivalence.

A small piece of unfinished business:

Higman and Scott worked both with Robinson games, and with Fraïssé games where the players take turns to extend finitely generated groups (rather than finite sets of equations and inequations).

They claimed to show, for both forms of game, that if P is any property then the second player has either a strategy for guaranteeing that the constructed group has P, or the same for not-P.

In *Journal of Algebra* 1993 I gave counterexamples to both results. (Nobody has ever cited this paper.)

Consider the Fraïssé-type. The heart of the refutation was a necessary and sufficient condition for the second player to have a strategy to enforce certain properties.

Namely, let F be the set of isomorphism types of finitely generated groups. Write $\mathcal{P}_\omega(F)$ for the set of all finite or countable subsets of F. For each $W \subseteq \mathcal{P}_\omega(F)$, let ϕ_W be the property that a group has if the set of its finitely generated subgroups is in W.

> The second player can enforce ϕ_W if and only if some subset of W is cofinal in $\mathcal{P}_\omega(F)$ and closed under union of chains.

Then we use a result of Ulam to find W so that neither ϕ_W nor $\phi_{\text{not-}W}$ is enforceable.

The proof of the necessary and sufficient condition uses the fact that groups form a Fraïssé class.
In particular they have the joint embedding property and the amalgamation property.

This raises the question: Find a necessary and sufficient condition in classes of algebras where these two properties fail. (Failure of amalgamation is what matters.)

Shelah told me he can answer the question, but I asked him to leave it with me until I give up. If I have any news in time for the Istanbul meeting I will report it.

4. Contact made

In 1981 in Jerusalem Greg Cherlin gave me a copy of Boris Zilber's 1977 registered manuscript, which contained early forms of many of his best-known insights.
That year I dictated a translation of this to Simon Thomas who wrote it up from his own point of view.
I believe Simon's text influenced the research of Ali Nesin and Alexandre Borovik among others.

In London we looked up work of Oleg and Boris in *Mat. Zametki* from the mid 1970s, and found a close parallel between their results on model-completeness and prime models and results of my student Michael Mortimer and Dan Saracino in the US. The main difference was that in the West we had been about 6 months slower.

On 14 April 1982 Oleg wrote to me in Russian:

'Dear Professor Hodges,
I got a copy of the book 'Theory of Models and its Applications' which you might like to have. I am sending it to you. I am also sending you the book 'Investigations in Theoretical Programming' … In turn I would like to have the book

> S. Shelah, Classification Theory and the Number of Nonisomorphic Models, North-Holland 1978.

If there is any other Russian mathematical literature that you want, write to me and I will try to help.
Sincerely Yours
O. Belegradek'

O. Belegradek, 'Classes of algebras with internal mappings', in *Investigations in Theoretical Programming*. Alma Ata 1981:

> Question. Given a finite group G and a map $\beta : G^n \to G$, is there always a finite group $H \supseteq G$ in which β is represented by a term with parameters?

In fact a positive answer is contained in Maurer and Rhodes (1965). But not knowing this, I showed that the answer is yes with a term that depends only on n. This is in the Proceedings of the Aachen 1983 conference on Models and Sets.

Oleg replied in 1986 by answering a question about Morley ranks at the end of my *Building Models by Games*.

In the early 1980s the East Berlin model theorists (led by Ingo Dahn) found they could hold conferences for logicians from both Eastern and Western Europe. These Easter Conferences were held every year from 1983 to 1991.

In 1986 Sergei Goncharov from Siberia attended.

In 1988 Oleg Belegradek and Boris Zilber attended; this was the first time I met them. Oleg attended also in 1989, 1990 and 1991.

Later in 1988 both Oleg and Boris attended a Durham Symposium that Otto Kegel, Peter Neumann and I organised. I remember that Higman was particularly eager to meet Oleg.

From 1991 people and information could move freely from East to West.

Oleg visited me in London, and I visited Oleg and Boris in Kemerovo.
But mathematically our paths were moving apart.

Kemerovo suddenly became famous in Britain when it was reported that a man had murdered several people and made them into meat pies which were sold at Kemerovo railway station.
I still can’t remember whether I ate a pie there.

5. Oleg and me — some brief data

The academic family tree

Karl Weierstrass

Nikolai Bugaev

Dimitri Egorov

Nikolai Luzin

Andrei Kolmogorov

Anatolii Mal'tsev

Mikhail Taitslin

OLEG BELEGRADEK

Georg Frobenius

Issai Schur

Richard Rado

Kenneth Gravett

John Crossley

WILFRID HODGES

Belegradek numbers

0. Oleg Belegradek

1. Kobi Peterzil

2. Anand Pillay

3. Wilfrid Hodges