Traditional versus modern logic

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Aristotle's procedure in logic (4th c. BC)

Step One: the raw argument

> *Salt solutions always conduct electricity.*
> *Nothing green conducts electricity.*
> *Therefore there are no green salt solutions.*

Step Two: setting out of terms

E : conducting electricity
S : salt solution
G : green

Step Three: the syllogistic rule

> *From 'E is true of all S' and 'E is true of no G' there follows 'G is true of no S'.*

This is the syllogistic rule known in the Middle Ages as Celarent.

The modern procedure

Step One: the raw argument

The barber shaves everyone who doesn't shave themselves. Therefore the barber shaves himself.

Step Two: translation scheme

Domain	:	people
Sxy	:	x shaves y
b	:	the barber

Step Three: sequent

$$\forall x\,(\neg Sxx \to Sbx) \;\vdash\; Sbb.$$

Step Four: formal proof of sequent

$$\dfrac{\dfrac{\dfrac{\cancel{\neg Sbb}^{1} \qquad \dfrac{\forall x\,(\neg Sxx \to Sbx)}{(\neg Sbb \to Sbb)}}{Sbb} \qquad \cancel{\neg Sbb}^{1}}{\bot}}{Sbb}\;1$$

(A derivation generated by the rules of Natural Deduction)

Myths about the history of logic

Many changes in culture since logic was first studied.
Historical record very patchy in parts.
Some factual mistakes about the history are inevitable.

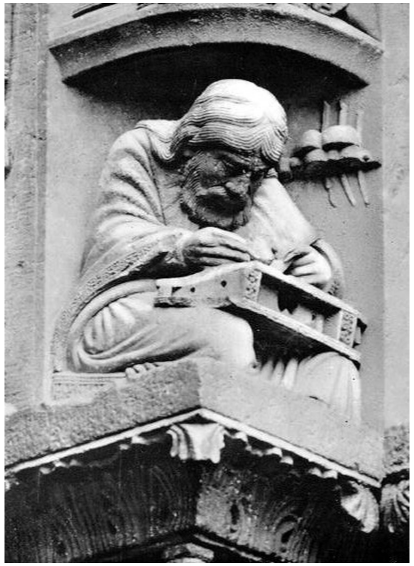
'Myths' : not just factual mistakes,
but mistaken attitudes that set up a smokescreen obscuring the facts.

We will look at three myths, all quoted from excellent textbooks.

Myth One: Logic comes partly from Pythagorean mathematics

> *It seems probable that the notion of demonstration attracted attention first in connexion with geometry. . . .*
> *The systematic study of [the science of geometry] seems to have begun in the Pythagorean school. . . .*
> *It is therefore safe to say that the ideal of a deductive system was known in the Pythagorean school and in the Platonic Academy, which continued some of its traditions.*
> *. . . It is, therefore, reasonable to suppose that one trend in Greek logic was determined in large part by reflection on the problems of presenting geometry as a deductive science.*

(William and Martha Kneale, *The Development of Logic*, Clarendon Press, Oxford 1962, pp. 2–6.)



Pythagoras (6th c. BC) at Chartres Cathedral

First difficulty: Aristotle denies it.

> *In the case of all discoveries the results of previous labours that have been handed down from others have been advanced bit by bit by those who have taken them on, whereas the original discoveries generally make an advance that is small at first, though much more useful than the development which later springs out of them. . . .*
> *Of the present inquiry, on the other hand, it was not the case that part of the work had been thoroughly done before, while part had not. Nothing existed at all.*

(Aristotle, *Sophistical Refutations*, in *The Complete Works of Aristotle, Vol. One*, ed. Jonathan Barnes, Princeton University Press 1984, pp. 313f.)

Second difficulty: A very serious mismatch between mathematical arguments and Aristotle's logic.

> *I have counted the assertions in the first ten propositions of [Euclid's] Elements III and the first five propositions of Elements VI. Of a total of 276 assertions, 199 assert relations, while 77 assert single-place predicates. . . .*
> *The majority of assertions is that of equivalences.*

(Reviel Netz, *The Shaping of Deduction in Greek Mathematics*, Cambridge University Press 1999, p. 197)

The first known attempts to reconcile Aristotle's logic with mathematics are Galen and Alexander of Aphrodisias in 2nd c. AD. They both concentrate on handling relations. We will come back to this.

Source of the myth:

Early propaganda wars between followers of Plato and Aristotle. 'Plato was better because he got it from Pythagoras who got it from divine sources.'

> *Next [Pythagoras] explained the whole of physics;*
> *he perfected both ethics and logic; he passed on all kinds of learning and science. Everything which has become part of human knowledge on any subject is fully dealt with in these writings.*

(Iamblichus, c. 300 AD, *On the Pythagorean life*, trans. Gillian Clark, Liverpool University Press 1989, p. 71.)

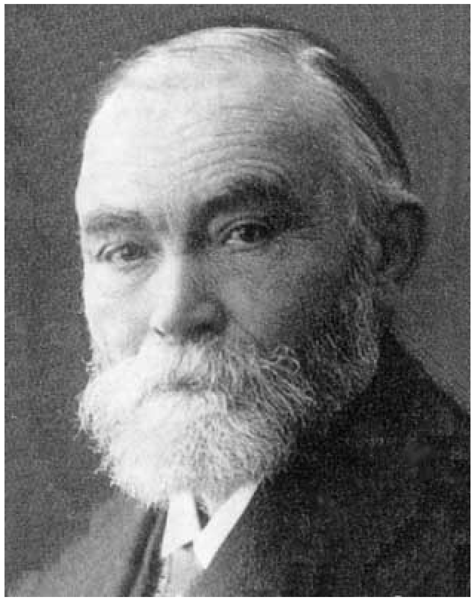
For historical evidence on the propaganda wars, see Burkert, *Lore and Science in Ancient Pythagoreanism*, Cambridge MA 1962.

Myth Two: Frege invented mathematical logic

> *The field of mathematical logic has its origin in Frege's new logic.*

(Joan Weiner, *Frege Explained: From Arithmetic to Analytic Philosophy*, Open Court, Chicago 2004, p. 163)

The implication is that Frege was the main origin of the switch from traditional aristotelian logic to modern mathematical logic in the period 1850–1900.



Gottlob Frege (1848–1925)

Difficulty: Giuseppe Peano (c. 1890) coined the name 'mathematical logic' and devised a programme for it.

> *Mr G. Frege, a professor at the University of Jena, to whom many interesting works in mathematical logic are due, of which the first dates from 1879, has arrived in turn, and by a quite independent path, in his book Grundgesetze der Arithmetik (1893) at the expression in symbols of a series of propositions concerning the concept of number. … The works of Frege are independent of those of the numerous authors of mathematical logic.*

(*Selected Works of Giuseppe Peano*, ed. Hubert C. Kennedy, George Allen and Unwin, London 1973, p. 191f.)

Peano lists the authors he used: Boole, Schröder, Peirce, Jevons, MacColl, Grassmann and Dedekind.

Giuseppe Peano (1858–1932)

Technical advances often credited to Frege:

Quantifiers. In fact Frege did have universal quantifiers (not existential), but the universal and existential quantifiers in use today trace back to Peirce via Schröder and Peano.

Justification of inductive definitions. Yes, he did this, but so did Dedekind in more influential work.

Russell's Paradox (based on Frege). True, though the paradox was discovered about the same time and independent of Frege and Russell by Zermelo (as Husserl confirms).

Source of the myth:

Frege was a *much* deeper and *much* more rigorous thinker than any of his contemporaries in logic.
Today we recognise this.

The myth comes from confusing

- our assessments of the value of his ideas, and
- the facts about historical influence.

Also some of his ideas (e.g. analysis of concepts, logicism) are more significant for philosophy of logic than they are for logic.
A particular danger for philosophers studying history.

Myth Three: Logic consists of formal systems

> *The Aristotelian theory of the syllogism is a system of true propositions concerning the constants A, E, I, and O. True propositions of a deductive system I call theses. Almost all theses of the Aristotelian logic are implications, i.e. propositions of the form 'If α, then β'.*

(Jan Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon Press, Oxford 1951, p. 20.)

Note that 'system' here is apparently short for 'deductive system', and that 'the Aristotelian logic' is identified with 'the Aristotelian theory of the syllogism'.

Jan Łukasiewicz (1878–1956)

Difficulty: Refer back to our introduction.

The formal system of Aristotle consists of his syllogistic rules and his methods for generating them.

The formal system of Natural Deduction consists of the derivable sequents of first-order logic and the Natural Deduction rules for deriving them.

This completely misses out *analysis*:
the process of translating the natural language arguments into logical form.

Source of the myth:

Assuming earlier logicians had the same aims as we do today.

Łukasiewicz used the interests of his own time to define what counts as 'logic'.

Reconciling logic and mathematics

What was the real problem about matching Aristotle's logic with mathematical reasoning?

Michael Friedman, 'Kant's theory of geometry', *Philosophical Review* 94 (1985) 455-506: syllogistic logic is 'monadic' and hence unable to handle relations.

Presumably Friedman means that the letters E, S etc. in syllogistic rules are written with at most one variable. But this can't be the problem; the rules of Natural Deduction are 'monadic' in exactly this sense too. Thus

$$\frac{A^x_t}{\exists x A} \qquad\qquad \frac{\exists x A \quad \begin{matrix}(A^x_a)\\ B\end{matrix}}{B}$$

Possible objection: 'A', 'B' in the Natural Deduction rules represent formulas that may contain many variables.

Answer: Exactly the same holds for syllogistic rules too. Ibn Sīnā (11th century Persia) frequently makes this point. For example he says artificial restrictions on the expressions represented by letters lead to a 'This is a stone' logic that can't handle relations. (*Qiyās* 45.9–11)

Note: Friedman ignored the process of *translating from natural language to syllogistic forms*. This is exactly Łukasiewicz's oversight, and it leads to historical error. (But Friedman is no doubt right about Kant.)

In fact Ibn Sīnā had two ways of handling relations.

- He could have free variables or pronouns in sentences, e.g.
  'This line and that line are both parallel to line L'.
- He could talk about ordered pairs, e.g.
  'The pair of lines L,M is a pair of parallel lines'.

These are essentially the same devices that we use today. For example in Tarski semantics we use ordered n-tuples to make assignments to the variables of a formula.

So we still need to know: What was the problem about doing mathematics with syllogisms?

The real problem emerges if we follow in detail Ibn Sīnā's proof (*Qiyās* p. 59) of

C equals B. B equals D. Therefore C equals D.

We put brackets {, } to show the terms that need to be set out.

Step One (a propositional syllogism, not in Aristotle)

$$\frac{\{C \text{ equals } B\} \qquad\qquad \{B \text{ equals } D\}}{\{C \text{ equals } B\} \text{ and } \{B \text{ equals } D\}}$$

Step Two (paraphrase)

$$\frac{\{C \text{ equals } B\} \text{ and } \{B \text{ equals } D\}}{\{B\} \text{ is } \{\text{a thing that } C \text{ is equal to and is equal to } D\}}$$

Step Three (can be written as a syllogism)

$$\frac{\{B\} \text{ is } \{\text{a thing that } C \text{ is equal to and is equal to } D\}}{\text{Some } \{\text{thing}\} \text{ is } \{\text{a thing that } C \text{ is equal to and is equal to } D\}}$$

Step Four (paraphrase)
Write E for: 'equal to something that is equal to D'.

$$\frac{\text{Some } \{\text{thing}\} \text{ is } \{\text{a thing that } C \text{ is equal to and is equal to } D\}}{\{\text{C}\} \text{ is an } \{\text{E}\}}$$

Step Five (syllogism, uses axiom of equality)

$$\frac{\{\text{C}\} \text{ is an } \{\text{E}\} \qquad\qquad \text{Every } \{E\} \text{ is } \{\text{equal to } D\}}{\{\text{C}\} \text{ is } \{\text{equal to } D\}}$$

The argument is all valid, but two of the steps are paraphrases that *change the terms*.

> *[Our doubt about the analytic character of arithmetic] can only be canceled by means of a gapless chain of deductions, so that no step could appear in it that is not in accordance with one of a few inference principles that are recognized as purely logical. ...*
> *We cannot give too many warnings against the danger of confusing points of view and switching from one question to another, a danger to which we are particularly exposed because we are accustomed to thinking in some language or other and because grammar ... is a mixture of the logical and the psychological.*

Frege, *Grundlagen der Arithmetik* §90; *Logic* (unpub.)

This is exactly Frege's critique of earlier logic.

Before Frege and Peano, logicians justified arguments *one step at a time*, i.e. they *formalised locally*. They were happy to carry out paraphrases (what Leibniz called 'grammatical analyses') between the logical steps.

Peano's programme was to turn entire mathematical proofs into symbolic arguments, without ever stepping back into paraphrase.

Thanks to Peano, modern logicians choose symbols for an entire argument at once. Any switch between symbols is governed by logical operations on the symbols.

A few morals:

1. To compare traditional aristotelian logic and modern mathematical logic, you have to be aware of the step of translating from natural language to symbols. The 'formal system' approach to logic hides this.
2. The aims of traditional logic at the translation step were different from ours today, in important ways that Frege correctly diagnosed.
3. To get the history of logic right, you have to *read earlier logicians*, slowly and many times over. You can't do it by having beautiful ideas.
4. What people tell you about the history of logic is generally wrong. If you're interested, please check it out for yourself and tell us.