Reconciling Greek mathematics and Greek logic - Galen's question and Ibn Sina's answer

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This will be a talk in the history of logic. Some dates:

- **Aristotle** Greece, 384–322 BC.
- **Chrysippus** Greece, c. 279–c. 206 BC.
- **Galen** Rome and Eastern Roman Empire, AD 129–c. 200.
- **Alexander of Aphrodisias** Greece, around late 2nd c. AD.
- **Ibn Sina** Persia, AD 980–1037.

> We have sometimes seen syllogistic discourse in which a proof is devised which has a single conclusion but more than two premises in it.
> There are demonstrations of this kind in the Book of Elements in geometry, and elsewhere.
> (Ibn Sīnā, *Qiyās* 433.6–8)

So Ibn Sīnā thought Euclid wrote 'syllogistic discourse'.

But here's the problem:
Nothing that survives of Euclid looks anything like syllogisms.

Classical Greek mathematics proceeded by deduction.
But see the texts of Hippocrates (before Aristotle) and Autolycus (slightly after Aristotle), H (= Handout) 1.1.

- Both use letters, but differently from anything in Aristotle's syllogisms (though related things are in Aristotle's *justifications* of the second and third figure syllogisms).
- The steps from one statement to another are mathematical moves, not logical ones.

Hardly believable that Aristotle reached his syllogisms by analysing mathematical arguments like these.

But the problem is more pressing than just to identify Aristotle's sources.
In logic we study and classify the forms of valid arguments.
If we can't identify the forms of the arguments in Euclid of all people, what credibility do we have?

Some logicians have held that there are some specifically mathematical rules of argument.
This view is said to have been Platonist,
and rejected by the Aristotelians (Peripatetics).
It does frankly look like a cop-out.

The Peripatetic view:

> Every deduction is formed through one or other of [the three syllogistic] figures.
> (*Prior Analytics* i.23–36)

Aristotle justifies this only by showing that some already identified argument forms can be seen to involve syllogisms.

C. S. Peirce (1898), fresh from inventing first-order logic, proves that in a sense every deduction 'is' a syllogism in mood Barbara. (See H1.12.)

Moral:

Aristotle's claim is not stupid.
Rather, it is vacuous until we explain 'is'.

Slogan:

In history of logic, a formalism without the instruction manual conveys nothing.

A logician's explanation of the reasoning in a text T, as we understand logic, has two components:

- A collection P of argument patterns ('moods', 'inference rules').
- Criteria for determining whether or not the arguments in T are instances of the patterns in P.

The Handout (H2) describes four sets of argument patterns:

(1) Aristotle's syllogisms.
(2) The Stoic propositional syllogisms.
(3) Patterns recognised by Ibn Sīnā.
(4) Natural deduction (Prawitz).

The ‘multiple generality’ red herring:

> ‘Syllogistic cannot handle arguments with multiple generality.’

This is from Jim Hankinson 1993, but similar statements are in Jonathan Barnes, Michael Friedman, Johan van Benthem among others.
I can’t trace it any earlier than the 1970s.

Given the spirit of the times, I guess the statement rests on some *formal* property of the syllogistic patterns, probably that they had just one variable.
But then this is a non sequitur.

A syllogism is a *single step* of reasoning.
So in comparison with modern systems of logic, ‘syllogism’ corresponds to ‘rule of inference’.
More precisely, a predicative syllogism is (an instance of) a rule of inference for relativised quantifiers.

But also in virtually all modern logical systems,
each rule of inference involves at most one quantified variable.

For example here are Prawitz’s rules of inference for classical natural deduction (H2.4):

&I) $\dfrac{A \quad B}{A \& B}$ &E) $\dfrac{A \& B}{A} \quad \dfrac{A \& B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \overset{(A)}{C} \quad \overset{(B)}{C}}{C}$

⊃I) $\dfrac{\overset{(A)}{B}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \overset{(A^x_a)}{B}}{B}$

$\curlywedge_I$) $\dfrac{\curlywedge}{A}$ $\curlywedge_C$) $\dfrac{\overset{(\sim A)}{\curlywedge}}{A}$

So the issue is what meanings can be given to the syllogistic letters — a question for the instruction manual.

For Aristotle the answer is a little unclear, but he tended to put names of natural kinds,
perhaps because of the precursors of syllogisms in Platonic discussions.

Ibn Sīnā very definitely expected the syllogistic letters to stand for meanings with parameters,
like [FATHER OF x] and [EQUAL TO y].

Having cleared away that irrelevance,
we return to history.

Aristotle's *Posterior Analytics* relates to the overall structure of mathematics as a deductive axiomatic science, but says virtually nothing about the proof rules.

Between Aristotle and mid 2nd century AD, we have only fragments and second-hand reports of logic,
little of it directly relevant to mathematical reasoning.

Mid 2nd c. AD, Galen in his *Institutio Logica* describes a class of arguments which he calls 'relational arguments'. (See H1.2.)

A relational argument

(a) is about relations between two or more things (hence 'relational');
(b) rests on an 'axiom' which expresses a self-evident conceptual (*tēi noēsei*) truth;
(c) the axiom states a 'universal' property of the relation.

Typical example, cited by Galen from Euclid *Elements* Prop. 1 (trans. Heath):

> Each of the straight lines CA, CB is equal to AB.
> And things which are equal to the same thing are also equal to one another;
> therefore CA is also equal to CB.

Galen says also that relational arguments are particularly common in arithmetic.

Alexander of Aphrodisias (Commentary on *Prior Analytics*, c. AD 200),
noting that Galen's 'axiom' talks of two or more things, observes that the quantifiers can be amalgamated into a single quantifier over ordered tuples. (See H1.3.)
This rests on the fact that the quantifiers are all universal, but Alexander may not have been aware of that.

Since this device first appears in Alexander, I call it *Alexander's device*.
It was copied right up to John Stuart Mill, *A System of Logic* ii.4.4.

It may be what Galen has in mind when he says we can 'reduce' relational arguments to predicative syllogisms.

Ibn Sīnā, broadly following Alexander, gives the following proof for transitivity of equivalence of lines.

Since the proof is syllogistic, Ibn Sīnā needs to indicate the subject and predicate of each sentence. He has a convention for doing this, but it doesn't translate. So instead I use curly brackets.

The proof is copied at H1.6. We will go through its features in a series of numbered points.

$$\dfrac{\{C \equiv B\} \qquad \{B \equiv D\}}{\{C \equiv B\} \text{ and } \{B \equiv D\}}\ (\alpha)$$

$$\Big|\ (\beta)$$

$$\dfrac{\{B\}\ \{\text{has } C \equiv \text{it and is} \equiv D\}}{\text{Some } \{\text{line}\}\ \{\text{has } C \equiv \text{it and is} \equiv D\}}\ (\gamma)$$

$$\Big|\ (\delta)$$

$$\dfrac{\begin{array}{l}\{C, D\} \text{ is a } \{\text{pair of lines with} \\ \text{some line} \equiv\text{-between them}\}\end{array} \quad \begin{array}{l}\text{Every } \{\text{pair of lines with some line} \equiv\text{-} \\ \text{between them}\} \text{ is a } \{\text{pair of} \equiv \text{lines}\}\end{array}}{\{C, D\} \text{ is a } \{\text{pair of} \equiv \text{lines}\}}\ (\varepsilon)$$

$$\Big|\ (\zeta)$$

$$C \equiv D$$

There are three separate subarguments, with connecting links.

Counting the terms in a complex argument, Ibn Sīnā takes the subarguments separately, and distinguishes between the conclusion of a step and the same proposition as premise of the next step. (See H1.7.)

So it seems likely that he pictured complex arguments as in our diagram.

In any case Ibn Sīnā, like all logicians before the mid 19th century, used logic to validate *each inference step separately*. Quite different formalisms can be used for two consecutive steps.
In other words, the formalising is *local*.
(Cf. H1.9 and Hodges, 'Traditional logic, modern logic and natural language', *J. Phil. Logic* 38 (2009) 589–606.)

Incidentally this prevents making of assumptions that are discharged several steps later (as often in Euclid).
We'll see below how Ibn Sīnā gets around this.

We take first the subarguments, then the connecting links.

It will be helpful to classify the status of each step at one of four levels (for a particular logician, of course):

- (Syl) Recognised as a syllogism, i.e. as an instance of a recognised syllogistic form.
- (Inf) Recognised as an inference step but not as a syllogism.
- (Rec) Recognised, but not as an inference step.
- (NR) Not recognised.

For Ibn Sīnā, step (ε) has status (Syl).
It's an instance of the singular form of the predicative syllogism Barbara (see H2.1).

Note how Alexander's device was needed in order to bring it into this form.
Alexander also recognised it as a syllogism.

Step (γ) was perhaps not recognised by Alexander.
He hides it inside another step (see H1.3).

For Ibn Sīnā it probably has status at least (Rec), since he seems to state it separately.
He is followed in this by his student Bahmanyār.

For Ibn Sīnā, single-premise steps are never syllogisms unless there is a second hidden premise.
But does Ibn Sīnā regard this step as an inference at all?

'Line' seems to appear from nowhere in the conclusion.
But for Ibn Sīnā, B is an indeterminate individual essence containing the meaning [LINE] as one of its constituents.
Peirce achieved the same effect by using sortal variables; take B as a variable in the sort 'line'.

So perhaps Ibn Sīnā regards the meaning [SOME LINE] as got by stripping away from the concept B the parts that identify a particular line. If so, it's hardly an inference.

Contrast Sextus Empiricus. We assume he has the rule (∃I) in mind when he cites the argument

> If a god has said to you that this man will be rich, this man will be rich; but this god has said to you that this man will be rich; therefore he will be rich.

until we see his comment (H1.4):

> we assent to the conclusion not so much on account of the logical force of the premisses as because of our belief in the statement of the god. (!!)

(Unless he is doing a rather subtle irony? I doubt it.)
I put the step down as (NR) for him.

For Ibn Sīnā, (α) is certainly a recognised move.
The conjunction of two descriptive meanings [X], [Y] is a single meaning whose criterion of satisfaction is the intersection of the criteria of [X] and [Y].
In our case the descriptive meanings are propositional.

But for Ibn Sīnā, forming the conjunction is distinct from inferring it from premises.
He may have thought that assenting to the conjunction is the same thing as assenting to both conjuncts;
in which case there is no inference.
So tentatively I rate this (Rec).

There remain the connecting links (β), (δ), (ζ).
These were needed to bring the premises and conclusions to the required forms for carrying out the other steps.
I call such moves *paraphrases*.

I believe that they are the main things described by Leibniz as *non-syllogistic steps* or *grammatical analyses*, and by Frege as *changes of viewpoint*.

Before local formalising was abandoned (Frege, Peano), logicians saw no need to justify these steps, since they preserve meaning.

Paraphrases were discussed briefly by Aristotle, and apparently more fully by Stoics in the early Roman Empire.

Difference of terminology:
For Aristotelians a 'syllogism' includes any paraphrasing of its premises or conclusion.
For the later Stoics the paraphrase is outside the syllogism; paraphrase plus syllogism constitute a 'subsyllogistic' argument.
(See Alexander and Al-Fārābī, H1.5.)

Some later medieval logicians (e.g. Razi in Arabic, Ockham in Latin) tried to find new syllogistic moods that would apply directly to the unparaphrased sentences.
(See El-Rouayheb, *Relational Syllogisms and the History of Arabic Logic, 900–1900*, Brill 2010.)
If successful, this would have generated a kind of natural language logic.

It would still be local formalising.
So for example we might need to choose new terms at each switch of step.

Frege's diagnosis

The main reason for the paraphrases is that the rules of syllogism are required to act at particular grammatical sites in the sentence (e.g. the subject).
The rules should be rewritten to apply wherever we choose.

In particular we should be able to apply the rules to terms at *any syntactic depth* in a formula.
We should be able to separate out a function argument at any level.

(See H1.10 and the second diagram in H1.6.)

I don't think Ibn Sīnā made the same analysis as Frege.
But one theory of his led to something remarkably similar.

To Ibn Sīnā, a sentence has a syllogistic sentence at its core,
but we nearly always mean, and often express, various 'additions' (*ziyādāt*) in the form of conditions, modalities, extra function arguments, etc. etc.

If core sentences make a valid inference step,
the step often remains valid after we add the *ziyādāt*.

Ibn Sīnā's Rule (in standard first-order logic):

> Let T be a set of formulas and ϕ, ψ formulas. Let $\delta(p)$ be a formula in which p occurs only positively, and p is not in the scope of any quantifier on a variable free in some formula of T. Suppose
>
> $$T, \phi \ \vdash \ \psi.$$
>
> Then
>
> $$T, \delta(\phi) \ \vdash \ \delta(\psi).$$

An example from Ibn Sīnā: If $\phi, \psi \vdash \chi$, then

$$\phi,\ \text{'If } \theta \text{ then } \psi\text{'} \ \vdash \ \text{'If } \theta \text{ then } \chi\text{'}.$$

In H2.3 I assemble various logical devices that seem to have been known to Ibn Sīnā.
To confirm his assumption that he can handle anything in Euclid by his syllogisms,
I present the devices in the form of a formal first-order calculus $\mathcal{IS}$ and prove its completeness.
I make four remarks about this system.

Remark One For simplicity I didn't include the predicative syllogisms. These serve to regulate the quantifiers, which for Ibn Sīnā are always relativised to a predicate. But relativised quantifiers can be handled within first-order logic.

Remark Two But I do include Chrysippus' fifth indemonstrable, in both forms noted by Ibn Sīnā.
This plays a role close to modus ponens. (See H2.2.)

I also include Ibn Sīnā's Rule. This mimics some key ingredients of a natural deduction calculus. The following is due to Ibn Sīnā himself, and it removes the need for assumptions that are later discharged.

Given $T, \phi \vdash \psi$ we can prove $T \vdash (\phi \to \psi)$ as follows:

By Ibn Sīnā's Rule, $T, (\phi \to \phi) \vdash (\phi \to \psi)$.
But $(\phi \to \phi)$ is an axiom, so $T \vdash (\phi \to \psi)$.

Likewise we mimic the rule (∃E):

Suppose $T, \phi(x) \vdash \psi$ where x is not free in ψ or any formula of T.
Then by Ibn Sīnā's Rule, $T, \exists x\phi(x) \vdash \exists x\psi$,
so $T, \exists x\phi(x) \vdash \psi$ since x is not free in ψ.

(I haven't seen such an argument in Ibn Sīnā himself.)

Remark Three Ibn Sīnā was steadfastly against regimenting Arabic for logical purposes.
He thought it blinds logicians to the kinds of thing that happen in actual scientific language,
and in particular to the kinds of *ziyāda* illustrated above.

Remark Four Recall again that Ibn Sīnā formalised locally.
He wouldn't have seen the point of having a calculus in which we can validate an entire argument from Euclid,
as opposed to validating each of its steps.

A final remark on Alexander's device of ordered pairs.
Ian Mueller in his edition of Alexander's commentary on *Prior Analytics* comments:

> The development of the logic of relations in the nineteenth century has made clear that Alexander is barking up the wrong tree here.

I very much doubt that Peirce would have agreed.
See H1.11, where the use of ordered pairs leads Peirce to discover first-order logic.

With hindsight the tipping point was not just the use of cartesian powers of the universe, but the use of *mixed universal and existential quantifiers*.

Ibn Sīnā came on these through his *ziyādāt*.
There are various reasons why he didn't react like Peirce.

But just suppose Galen had noticed mixed quantifiers and not stuck to universal ones!