Handout for: Reconciling Greek mathematics and Greek logic - Galen's question and Ibn Sina's answer

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1 Texts

1.1 Greek mathematics of Aristotle's time

Hippocrates of Chios (3rd quarter of 5th c. BC), surviving in Simplicius' (6th c. AD) quotation of Aristotle's pupil Eudemus. The translation is by Ivor Bulmer-Thomas in the Loeb volume *Greek Mathematical Works 1* (1939), pp. 237–243.

I shall set out what Eudemus wrote word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements* on account of the summary style of Eudemus, who set out his proofs in abridged form in conformity with the ancient practice. . . .

The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases. And this he proved by showing that the squares on the diameters have the same ratios as the circles.

Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides. Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking a semicircle as the outer circumference of the lune, Hippocrates readily squared the lune. Next . . .

If [the outer circumference] were less than a semicircle, Hippocrates solved this also, using the following preliminary construction. Let there be a circle with diameter AB and centre K. . . .

Autolycus of Pitánē (end of 4th c. BC), our earliest surviving Greek mathematical text, though it seems to be exposition rather than original research. My translation from the Budé text ed. Germaine Aujac, *Autolycos de Pitane: La sphère en mouvement; levers et couchers héliaques*, Les Belles Lettres, Paris 2002, pp. 42–44.

If a sphere rotates evenly around its axis, all the points on the surface of the sphere and not on the axis will draw circles which are parallel and have the same poles on the sphere, and are orthogonal to the axis.

Let the sphere have axis the straight line AB with the points A and B as the poles, and let it rotate evenly around its axis AB. I say that all the points on the surface of the sphere and not on the axis will draw circles which are parallel and have the same poles on the sphere, and are orthogonal to the axis.

For let a point C be taken on the sphere. And let a perpendicular line CD be taken from C to the line AB. And let a plane be drawn through the points A, B and the line CD; it will cut [the sphere] in a circle. Let ACB be the semicircle of this circle. If the semicircle ACB is carried around, up to its initial position, while holding the line AB fixed, the line CD too will be carried around with the movement of the semicircle ACB while remaining perpendicular to AB, and it will draw a circle in the sphere, with centre the point D, and the radius is CD which is perpendicular to the axis AB.

And it is clear that the points A, B will be poles of the drawn circle, since AB is a perpendicular taken from the centre of the sphere and extended to the surface of the sphere.

Likewise we will show that all the points on the surface of the sphere which are not on the axis will draw circles that are orthogonal to the axis AB with the same poles on the sphere. And circles around the same poles on a sphere are parallel.

So all the points on the surface of the sphere and not on the axis will draw circles that are parallel and have the same poles in the sphere, and are orthogonal to the axis.

1.2 Galen on relational arguments

Galen, *Institutio Logica* xvi. My translation, but the state of the text is so bad that any translation should be taken with a grain of salt.

(1) There is another third kind of syllogism useful for demonstrations, which I describe as 'relational' — though the Aristotelians insist on counting them among the predicative syllogisms. They are used not a little by sceptics and theoretical and applied arithmeticians for arguments such as: Theo owns twice as much as Dio. But Philo owns twice as much as Theo. So Philo owns four times as much as Dio.

...

(5) As I said, there are a large number of syllogisms of this kind in theoretical and practical arithmetic, which all have it in common that they take their structure from some self-evident truths. Bearing these truths in mind in the phrases as spoken, we will be able to reduce such syllogisms to predicative ones, starting over again in a way that is clearer for us.

(6) Since it's a self-evident universal axiom that 'things equal to the same thing are also equal to each other', one can reason and demonstrate just as Euclid in his first Theorem made a demonstration showing that the sides of the triangle are equal. For since things equal to the same thing are also equal to each other, and it is demonstrated that the first and the second are equal to the third, the first will be equal to each of them.

...

(11) This syllogism is put in hypothetical form: 'If Socrates is son of

Sophroniscus, then Sophroniscus is father of Socrates; but Socrates is son of Sophroniscus; so Sophroniscus is father of Socrates.' In predicative premises the construction of the calculation will be more forceful, of course putting a universal sentence in front and hence some such axiom, 'The person that someone has as father, he is son of that person. Lamprocles has Socrates as father; then Lamprocles is son of Socrates'.

1.3 Alexander on relational arguments

Alexander of Aphrodisias, commentary on Aristotle, *Prior Analytics* i.32; trans. I. Mueller, *Alexander of Aphrodisias, On Aristotle Prior Analytics 1.32–46*, Duckworth, London 2006, p. 28.

Here he indicates clearly to us that one should not simply attend to the conclusion and think that there is a syllogism if something follows necessarily from what is assumed. For it is not the case that if a syllogism proves something by necessity thereby also where something is proved to follow by necessity from what is assumed, this is a syllogism, since necessity is more inclusive than syllogism. Accordingly, if it is not the case that if it follows by necessity from the assumption that A is equal to B and C to B that A is also equal to C, that this is thereby a syllogism. There will be a syllogistic inference if we assume in addition a universal premiss which says that things equal to the same thing are also equal to each other and we draw together what were taken as two premisses into one premiss equivalent to the two. This premiss is 'A and C are equal to the same thing (since they are equal to B)'. In this way it follows syllogistically that A and C are equal to each other.

Similar to this is thinking that one proves syllogistically that A is greater than C if one assumes that A is greater than B and B is greater than C, on the grounds that this conclusion does follow necessarily. But this is not in itself a syllogism unless the universal premiss 'Everything which is greater than what is greater than something is also greater than what is less than that' is assumed in addition and the two things assumed are made into one premiss — the minor in the syllogism — which says that A is greater than B, which is greater than C. For in this way it will follow syllogistically that A is also greater than C.

1.4 Sextus Empiricus on existential quantifier rules

Sextus Empiricus, *Outlines of Pyrrhonism* ii.141, Loeb edition, trans. R. G. Bury

And of arguments which deduce something non-evident, some conduct us through the premisses to the conclusion by way of progression only, others both by way of progression and by way of discovery as well. By progression, for instance, are those which seem to depend on belief and memory, such as the argument "If a god has said to you that this man will be rich, this man will be rich; but this god (assume that I point to Zeus) has said to you that this man will be rich; therefore he will be rich"; for we assent to the conclusion not so much on account of the logical force of the premisses as because of our belief in the statement of the god.

1.5 Contrasting notions of syllogism

Alexander of Aphrodisias, *On Aristotle Prior Analytics* 84, 12–16 (trans. Barnes et al.)

This is an argument of the sort which the more recent thinkers call subsyllogistic; it takes something equivalent to the syllogistic premiss and deduces the same thing from it. ('Does not hold of some' has been transformed into 'does not hold of every', which is equivalent to it.) The more recent thinkers deny that such arguments are syllogisms, since they look to the words and the expression. Aristotle, however, looks to the meanings (when the same things are meant) rather than to the words and says that the same syllogism is deduced when the expression of the conclusion is transformed in this way — granted that the conjunction is in general syllogistic.

Al-Fārābī, *Book of the Syllogism* 78A (my translation)

If a statement is not in the form of one of the [syllogistic moods] given above, but can be brought to one of those forms by adding things or removing things or rearranging the order, without affecting the original sense of the statement, then this statement is a syllogism.

1.6 Ibn Sīnā on transitivity of equivalence

Ibn Sīnā *Qiyās* i.6 and his student Bahmanyār ibn al-Marzubān on transitivity of equality. The translation of Ibn Sīnā is mine; that of Bahmanyār is slightly adjusted from Khaled el-Rouayheb, *Relational Syllogisms and the History of Arabic Logic, 900–1900*, Brill 2010, p. 24.

(Ibn Sīnā) Thus when you say

$$C \text{ is equal to } B \text{ and } B \text{ is equal to } D\text{, so } C \text{ is equal to } D. \tag{1}$$

this is complete for you only when you become aware that C is equal to something equal to D, and that a thing that is equal to something equal [to X] is itself equal [to X].

(Bahmanyār) If you say 'C is equal to B and B is equal to D, therefore C is equal to D' you have not made explicit one of the two premises. For the form of the syllogism to be a complete syllogism we say 'C is something that is equal to B which is something that is equal to D, and anything equal to something equal [to X] is equal [to X], and so C is equal to D'. This is as if someone said, 'C is equal to something equal to D, and everything equal to something equal to D is equal to D, so C is equal to D'.

$$\dfrac{\{C \equiv B\} \qquad \{B \equiv D\}}{\{C \equiv B\} \text{ and } \{B \equiv D\}}\,(\alpha)$$

$$\Big|\;(\beta)$$

$$\dfrac{\{B\}\ \{\text{has } C \equiv \text{ it and is } \equiv D\}}{\text{Some } \{\text{line}\}\ \{\text{has } C \equiv \text{ it and is } \equiv D\}}\,(\gamma)$$

$$\Big|\;(\delta)$$

$$\dfrac{\begin{array}{l}\{C, D\} \text{ is a } \{\text{pair of lines with} \\ \text{some line } \equiv\text{-between them}\}\end{array} \qquad \begin{array}{l}\text{Every } \{\text{pair of lines with some line } \equiv\text{-} \\ \text{between them}\} \text{ is a } \{\text{pair of } \equiv \text{ lines}\}\end{array}}{\{C, D\} \text{ is a } \{\text{pair of } \equiv \text{ lines}\}}\,(\varepsilon)$$

$$\Big|\;(\zeta)$$

$$C \equiv D$$

Below is the previous diagram translated into first-order logic. Comparing the two diagrams, we see that (β) identifies the subject in $x \equiv z \wedge z \equiv y$ as z, then (δ) reidentifies the subject as (x, y), and finally (ζ) removes the identification of the subject in $x \equiv y$ as (x, y). Instead of bringing the subject out to subject position at the head of the formula, the second diagram applies rules at whatever place in the formula is appropriate.

$$\dfrac{\dfrac{\dfrac{C \equiv B \qquad B \equiv D}{(C \equiv B \wedge B \equiv D)}\,(\alpha)}{\exists z(C \equiv z \wedge z \equiv D)}\,(\gamma) \qquad \forall x \forall y(\exists z(x \equiv z \wedge z \equiv y) \rightarrow x \equiv y)}{C \equiv D}\,(\varepsilon)$$

1.7 Ibn Sīnā on compound syllogisms

From Ibn Sīnā *Qiyās* ix.3, on counting the size of a compound syllogism (my translation).

In the case where the other [proximate premise] has to be derived [as well], a syllogism with two premises is introduced in order to derive it. Then at one level there are four premises and two conclusions, and at the second level there are two premises and a single conclusion. So the compound [syllogism] contains six premises altogether and three conclusions altogether. The number of conclusions is half the number of premises. Each of the [simple] syllogisms contains three terms and a conclusion. Suppose in fact that each [proximate] premise [is proved by] a syllogism, and the two [proximate] premises share a term. Then there are six terms, except that one of them is shared in the middle, so there are five terms. The shared term and the term at one end of the five give rise to one proximate premise, and the shared term and the other end term give rise to the other [proximate] premise. The two end terms of the five give rise to the goal which is the target of the compound syllogism.

1.8 Ibn Sīnā on ordered pairs

Ibn Sīnā *Qiyās* ix.7 commenting on *Prior Analytics* i.33, my translation.

It is said:

> Zayd is Zayd the rich.

and

> Zayd the rich won't survive till tomorrow unless [his] ownership of riches survives.

So the combination of the two meanings ([ZAYD] and [THE RICH]) won't continue to be satisfied if one of the two meanings doesn't continue to be satisfied. In this example one has to understand that [ZAYD THE RICH] is also a universal [i.e. a relation rather than a constant]. This is because [ZAYD] describes only a single person, whereas the meaning [ZAYD THE RICH] can be true of many different things. And this is because Zayd the rich is a particular rich person with respect to a particular ownership of riches. We could find him an hour later still being Zayd, but no longer rich, so that he wouldn't still be Zayd the rich. And then [again] he could become Zayd the rich. But we wouldn't be referring [to the new ownership of riches] as numerically the same as the previous ownership of riches; it would be [a different exemplar] of the same species. So regarding him as Zayd, he is that same individual. But regarding him as the combination of Zayd and being rich, he is not numerically the same as the previous one. He would only be the same as the previous one if it was the same Zayd and numerically the same ownership of riches.

1.9 Ibn Sīnā on local validation

Ibn Sīnā *Autobiography*, trans. D. Gutas, *Avicenna and the Aristotelian Tradition*, Brill, Leiden 1988, pp. 27f. Gutas' 'Ascertained' is better translated 'verified', a near-synonym of 'validated'. Also Gutas explains 'conditions' (of the premises) as 'modalities'; but in Ibn Sīnā's normal usage the word (*šarṭ*) means side-conditions, particularly ones that the speaker intended but didn't say.

8. The next year and a half I devoted myself entirely to reading Philosophy: I read Logic and all the parts of philosophy once again. During this time I did not sleep completely through a single night, or occupy myself with anything else by day. I compiled a set of files for myself, and for

each argument that I examined, I recorded the syllogistic premisses it contained, the way in which they were composed, and the conclusions which they might yield, and I would also take into account the conditions of its premises until I had Ascertained that particular problem. ... So I continued until all the Philosophical Sciences became deeply rooted in me and I understood them as much as is humanly possible. Everything that I knew at that time is just as I know it now; I have added nothing more to it to this day.

9. Having mastered Logic, Physics and Mathematics, I ...

1.10 Frege on 'the main argument'

Gottlob Frege, *Begriffsschrift* §9, trans. in van Heijenoort, *From Frege to Gödel* pp 22f. Frege's italics.

If in an expression, whose content need not be capable of becoming a judgment, a simple or compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function.

...

In the mind of the speaker the subject is ordinarily the main argument (*hauptsächliche Argument*); the next in importance often appears as object. Through the choice between [grammatical] forms, such as active—passive, or between words, such as "heavier"—"lighter" and "give"—"receive", ordinary language is free to allow this or that component of the sentence to appear as main argument at will, a freedom that, however, is restricted by the scarcity of words.

1.11 Peirce on cartesian powers

C. S. Peirce, *Fragment on the Algebra of Logic*, 1884 (Volume v page 109f in the Indiana edition). This unpublished fragment seems to mark the moment when Peirce discovered first-order logic.

The first system of relationship which logic studies is that of an indefinite collection of units. It may be represented by the schema

$$|\,|\,|\,|\,|\,|\,|\,|\,|$$

These constitute the universe of discourse. Various names or conventional signs, for which letters may be used, are attached to these in various ways. The study of this schema gives rise to the Boolian calculus. The logic of relatives studies a collection of units arranged in an n-dimensional block, thus:

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Distinguishing the different dimensions by the letters $i, j, k,$ etc. we may write

$\Pi_i a_i$	for every i has the mark a.
$\Sigma_i a_i$	for some i has the mark a.
$\Pi_i \Pi_j \ell$	Every i is in the relation ℓ to every j.
$\Sigma_i \Pi_j \ell$	Some i is in the relation ℓ to all j's.

...

The logic of relatives, so understood, coincides with Professor Mitchell's multidimensional logic; and the logic of relatives as De Morgan and I have understood it, is a special case under this broader conception.

1.12 Peirce proving every inference is in Barbara

C. S. Peirce, *Reasoning and the Logic of Things*, Cambridge Lectures 1898, ed. K. E. Ketner, Harvard University Press, Cambridge Mass. 1992, pp. 131f. This passage is analysed in Wilfrid Hodges, 'The scope and limits of logic', in Dale Jacquette, *Philosophy of Logic*, Elsevier, Amsterdam 2007, pp. 44–46.

Suppose we draw a conclusion. Whether it be necessary or probable I do not care. Let **S** is **P** represent this conclusion. Now we certainly never can be warranted in drawing any conclusion about **S** from a premise, or set of premises, which does not relate in any way to **S**. If the inference is drawn from more than one premise, let all the premises be colligated into one copulative proposition. Then this single premise must relate to **S**; and in that sense, it may be represented thus: **S** is **M**. I do not, of course, mean that **S** need appear formally in this premise as a subject, far less as the sole subject. I only mean that "**S** is **M**" may in a general sense stand for any proposition which virtually relates to **S**. The inference, then, appears in this form

Premise	**S** is **M**
Conclusion	**S** is **P**

But, *whenever we draw a conclusion*, we have an idea, more or less definite, that the inference we are drawing is only an example of a whole class of possible inferences, in each of which from a premise more or less similar to the actual premise there would be a sound inference of a conclusion analogous to the actual conclusion. And not *only* is this idea present to our consciousness, — as is shown by our thinking that the premise *leads to* the conclusion, — but, what is still more important, there is a principle *actually operative* in the depths of our minds, — a *habit*, natural or acquired, by virtue of which we really *should* draw that analogous conclusion in each of those possible cases. This operative principle I call, after the logician Fries, the *leading principle* of the inference. But now *logic* supposes that reasonings are *criticised*; and as soon as the reasoner asks himself what *warrant* he has for concluding from **S** is **M** that **S** is **P**, he is driven to *formulate* his leading principle. Now in a very general sense we may write as representing that formulation, **M** is **P**. I write **M** is **P** instead of **P** is **M** because the inference takes place from **M** to **P**, that is **M** is antecedent while **P** is consequent. So that the reasoner in consequence of his self-criticism reforms his argument and substitutes in place of his original inference, this *complete* argument:

$$\begin{array}{ll} \textit{Premises} & \left\{ \begin{array}{l} \mathbf{M} \text{ is } \mathbf{P} \\ \mathbf{S} \text{ is } \mathbf{M} \end{array} \right. \\ \textit{Conclusion} & \mathbf{S} \text{ is } \mathbf{P} \end{array}$$

I do not mean that the formulation of the leading principle necessarily takes the form **M** is **P** in any *narrow sense*. I only mean that it must express some general relation between **M** is **P**, which not merely in reference to the special subject, **S**, but in all analogous cases will warrant the passage from

a premise similar to **S** is **M** to a conclusion analogous to **S** is **P**.

. . .

It is thus proved that *in an excessively general sense* every complete argument, i.e. every argument having a leading principle of maximum abstractness, is an argument in the form of *Barbara*.

2 Formal systems

2.1 The recombinant (*iqtirānī*) syllogistic moods

These are a generalisation of the predicative (= categorical) syllogistic moods of Aristotle, probably due to Ibn Sīnā. Aristotle's moods are listed on the left, Ibn Sīnā's additions on the right.

Ibn Sīnā follows the convention that in each proposition of a predicative syllogism, when the first term C is unsatisfied, then the proposition counts as false if it is affirmative, and true if it is negative. He should carry this convention over to the propositional moods on the right. But if he does this, and takes all propositions to be talking about the present (or about timeless truths), then the propositional moods collapse as shown in the singular moods below (using the formulas in brackets).

First figure, Barbara	
Every C is a B.	Whenever r, q.
Every B is an A.	Whenever q, p.
Then every C is an A.	Then whenever r, p.
Celarent	
Every C is a B.	Whenever r, q.
No B is an A.	Whenever q, not$p$.
Then no C is an A.	Then whenever r, not p.
Darii	
Some C is a B.	Sometimes r and q.
Every B is an A.	Whenever q, p.
Then some C is an A.	Then sometimes r and p.

Ferio
Some C is a B.
No B is an A.
Then every C is an A.

Sometimes r and q.
Whenever q, not p.
Then not always when r, p.

Second figure, Cesare
Every C is a B.
No A is an B.
Then no C is an A.

Whenever r, q.
Whenever p, not q.
Then whenever r, not p.

Camestres
No C is a B.
Every A is an B.
Then no C is an A.

Whenever r, not q.
Whenever q, p.
Then whenever r, not p.

Festino
Some C is a B.
No A is an B.
Then not every C is an A.

Sometimes r and q.
Whenever p, not q.
Then not always when r, p.

Baroco
Not every C is B.
Every A is an B.
Then not every C is an A.

Not always when r, q.
Whenever p, q.
Then not always when r, p.

Third figure, Darapti
Every B is a C.
Every B is an A.
Then some C is an A.

Whenever q, r.
Whenever q, p.
Then sometimes r and p.

Felapton
Every B is a C.
No B is an A.
Then not every C is an A.

Whenever q, r.
Whenever q, not p.
Then not always when r, p.

Datisi
Some B is a C.
Every B is an A.
Then some C is an A.

Sometimes q and r.
Whenever q, p.
Then sometimes r and p.

Disamis
Every B is a C.
Some B is an A.
Then some C is an A.

Whenever q, r.
Sometimes q and p.
Then sometimes r and p.

Bocardo
Every B is a C.
Not every B is an A.
Then not every C is an A.

Whenever q, r.
Not always when q, p.
Then not always when r, p.

Ferison
Some B is a C.
No B is an A.
Then not every C is an A.

Sometimes q and r.
Whenever q, not p.
Then not always r, p.

A term appearing only as subject can be a constant (e.g. the present moment in the propositional case). The resulting syllogism is said to be *singular*. The propositions containing this term lose their quantification; the Arabic convention was to count them as universally quantified. The resulting syllogistic moods are given below.

When the propositional moods are taken with time restricted to the present, all the propositions collapse. The resulting moods are the singular moods but with the remaining premise also reduced; I list the results in brackets in the righthand column below.

First figure, Barbara
C is a B.
Every B is an A.
C is an A.

$r \wedge q$.
Whenever q, p. $(q \wedge p)$
Then $r \wedge p$.

Celarent
C is a B.
No B is an A.
Then C is not an A.

$r \wedge q$.
Whenever q, not p. $(\neg(q \wedge p))$
Then $\neg(r \wedge p)$.

Second figure, Cesare
C is a B.
No A is an B.
Then C is not an A.

$r \wedge q$.
Whenever p, not q. $(\neg(p \wedge q))$
Then $\neg(r \wedge p)$.

Camestres	
C is not a B.	$\neg(r \wedge q)$.
Every A is an B.	Whenever q, p. $(q \wedge p)$)
Then C is not an A.	Then $\neg(r \wedge p)$.
Third figure, Darapti	
B is a C.	$q \wedge r$.
B is an A.	$q \wedge p$.
Then some C is an A.	Then sometimes $r \wedge p$. $(r \wedge p)$
Felapton	
B is a C.	$q \wedge r$.
B is not an A.	$\neg(q \wedge p)$.
Then not every C is an A.	Then not always when r, p. $(\neg(r \wedge p))$

2.2 The Stoic propositional syllogistic moods

The five indemonstrables of Chrysippus are as follows.

(1) If the first then the second.
The first.
Therefore the second.

(2) If the first then the second.
Not the second.
Therefore not the first.

(3) Not both the first and the second.
The first.
Therefore not the second.

(4) The first or the second.
The first.
Therefore not the second.

(5) The first or the second.
Not the first.
Therefore the second.

Ibn Sīnā, *Qiyās* 401.7 quotes (5) in his own words. He adds that we can also infer ϕ from 'ϕ or ψ' and 'Not ψ'.

2.3 The calculus $\mathcal{IS}$ (Ibn Sīnā)

We introduce a proof calculus $\mathcal{IS}$ for first order logic, and we sketch a proof of its completeness. The calculus is based on techniques known to Ibn Sīnā, but I stress straight away that he would never have combined them in this form.

The language is a standard first-order language with truth-functions $\neg, \wedge, \vee$, quantifier symbols $\forall, \exists$ and infinitely many variables, but no identity. We assume the signature is relational and at most countable. We allow ourselves to add new variables at will.

The calculus is presented in the form of a set of sequents $T \vdash \phi$, where ϕ is a formula and T is a set of formulas. Some basic sequents are given outright, and there are also derivation rules for deriving sequents from other sequents. We describe these sequents with symbols ϕ, ψ etc. as metavariables for formulas, and x, y etc. as metavariables for variables. The valid sequents are those generated from these rules.

We write $T, \phi \vdash \psi$ for $T \cup \{\phi\} \vdash \psi$, and similar things. If x and y are variables, we write $\phi[y/x]$ for the result of replacing all free occurrences of x in ϕ' by y, where ϕ' is the result of replacing all bound occurrences of y in ϕ by occurrences of another variable y' distinct from y and not occurring in ϕ.

Basic sequents

(Refl) $\phi \vdash \phi$ (NR)

(ExclM) $\vdash (\phi \vee \neg\phi)$

(NonC) $\vdash \neg(\phi \wedge \neg\phi)$

(ChrysL) $(\phi \vee \psi), \neg\phi \vdash \psi$ (Syl)

(ChrysR) $(\phi \vee \psi), \neg\psi \vdash \phi$ (Syl)

(DM$\wedge$) $\neg(\phi \wedge \psi) \vdash (\neg\phi \vee \neg\psi)$ (Inf)

(DM$\vee$) $\neg(\phi \vee \psi) \vdash (\neg\phi \wedge \neg\psi)$ (Inf)

(DM$\forall$) $\neg\forall x\phi \vdash \exists x\neg\phi$ (Inf)

(DM$\exists$) $\neg\exists x\phi \vdash \forall x\neg\phi$ (Inf)

($\wedge$I) $\phi, \psi \vdash (\phi \wedge \psi)$ (Rec?)

($\wedge$EL) $(\phi \wedge \psi) \vdash \phi$ (Rec?)

($\wedge$ER) $(\phi \wedge \psi) \vdash \psi$ (Rec?)

($\forall$E) If t is any variable, then $\forall x\phi \vdash \phi[t/x]$ (Inf?)

($\exists$I) If t is any variable, then $\phi[t/x] \vdash \exists x\phi$ (Inf?)

(Vac) If x doesn't occur free in ϕ then $\exists x\phi \vdash \phi$ (NR)

(Var) If y doesn't occur in ϕ then $\exists x\phi \vdash \exists y\phi[y/x]$ (NR)

Derivation rules

(Mono) If $T \vdash \psi$ then $T \cup U \vdash \psi$.

(Trans) If $T \vdash \psi$ and for each $\phi \in T$, $U \vdash \phi$, then $U \vdash \psi$.

(IS) (Ibn Sīnā's Rule) Suppose T a set of formulas and ϕ, ψ are formulas. Let $\delta(p)$ be a formula of IS but containing a propositional variable p which occurs only positively in $\delta(p)$ and doesn't occur in the scope of any quantifier on a variable free in some formula of T. If $T, \phi \vdash \psi$ then $T, \delta(\phi) \vdash \delta(\psi)$.

We begin with some structural properties not depending on any symbol of the language.

Lemma 1 *If $T \vdash \phi$ is valid, then $U \vdash \phi$ for some finite $U \subseteq T$.*

Proof This is true for each of the basic sequents, and is preserved by the derivation rules. □

Lemma 2 *If $\phi \in T$ then $T \vdash \phi$.*

Proof Suppose $\phi \in T$. By (Refl), $\phi \vdash \phi$, and so by (Mono), $T, \phi \vdash \phi$. But $T \cup \{\phi\} = T$. □

Lemma 3 *If $T, \phi \vdash \psi$ and $T \vdash \phi$, then $T \vdash \psi$.*

Proof By Lemma 2, $T \vdash \chi$ for each $\chi \in T$. By this and the assumption $T \vdash \phi$, we have $T \vdash \chi$ for each $\chi \in T \cup \{\phi\}$. So $T \vdash \psi$ by (Trans) and the assumption $T, \phi \vdash \psi$. □

The proofs above illustrate the use of the rules (Refl), (Mono) and (Trans). In future we will normally use them without mention.

Now follow some basic properties of $\neg$ and $\vee$.

Lemma 4 $\neg\neg\phi \vdash \phi$

Proof By (ChrysR), $(\phi \vee \neg\phi), \neg\neg\phi \vdash \phi$. Now apply Lemma 3 with (ExclM). □

Theorem 1 (Deduction Theorem) *(a) If $T, \phi \vdash \psi$ then $T \vdash (\psi \vee \neg\phi)$.*
(b) If $T, \neg\phi \vdash \psi$ then $T \vdash (\psi \vee \phi)$.

Proof (a) Assume $T, \phi \vdash \psi$. By Ibn Sīnā's Rule (IS), $T, (\phi \vee \neg\phi) \vdash (\psi \vee \neg\phi)$. By (ExclM), $T \vdash (\phi \vee \neg\phi)$. Now the result follows by Lemma 3.
(b) Assume $T, \neg\phi \vdash \psi$. Then by (a), $T \vdash (\psi \vee \neg\neg\phi)$. But by Lemma 4 and another application of Ibn Sīnā's Rule (IS), $(\psi \vee \neg\neg\phi) \vdash (\psi \vee \phi)$. □

Lemma 5 $\phi \vdash \neg\neg\phi$

Proof By ($\wedge$I), $\phi, \neg\phi \vdash (\phi \wedge \neg\phi)$, and so by the Deduction Theorem 1(a), $\phi \vdash ((\phi \wedge \neg\phi) \vee \neg\neg\phi)$. So by (NonC) and (ChrysL), $\phi \vdash \neg\neg\phi$. □

Lemma 6 *(a)* $(\phi \vee \neg\psi), \psi \vdash \phi$
(b) $(\neg\phi \vee \psi), \phi \vdash \psi$.

Proof (a) By Lemma 5, $\psi \vdash \neg\neg\psi$, and by (ChrysR), $(\phi \vee \neg\psi), \neg\neg\psi \vdash \phi$.
(b) Similar with (ChrsL). □

Lemma 7 $(\phi \wedge \neg\phi) \vee \psi \vdash \psi$

Proof By (Contr) and (ChrysL). □

Lemma 8 $(\phi \vee \psi) \vdash (\psi \vee \phi)$.

Proof By (ChrysL), $(\phi \vee \psi), \neg\phi \vdash \psi$. Then by the Deduction Theorem, Theorem 1(b), $(\phi \vee \psi) \vdash (\psi \vee \phi)$. □

Lemma 9 $\phi \vdash \phi \vee \psi$.

Proof By (Refl), $\phi, \neg\psi \vdash \phi$. Now apply the Deduction Theorem, Theorem 1(b). □

Lemma 10 $(\phi \wedge \neg\phi) \vdash \psi$

Proof By Lemma 9, $(\phi \wedge \neg\phi) \vdash ((\phi \wedge \neg\phi) \vee \psi)$. Now apply Lemma 7. □

Lemma 11 *(a) If $\phi \vdash \psi$ then $\neg\psi \vdash \neg\phi$.*
(b) If $\neg\psi \vdash \neg\phi$ then $\phi \vdash \psi$.

Proof (a) Assume $\phi \vdash \psi$. Then $(\phi \vee \neg\phi) \vdash (\psi \vee \neg\phi)$ by Ibn Sīnā's rule (IS), so $\vdash \psi \vee \neg\phi$ by (ExclM). Now use (ChrysL), $(\psi \vee \neg\phi), \neg\psi \vdash \neg\phi$.
(b) Similar, via $(\psi \vee \neg\psi) \vdash (\psi \vee \neg\phi)$ and Lemma 6(a). □

Some quantifier lemmas:

Lemma 12 *Suppose x doesn't occur free in ϕ. Then $\phi \vdash \forall x\phi$.*

Proof By Lemma 11(b) it suffices to prove $\neg\forall x\phi \vdash \neg\phi$ under the same hypothesis. Now $\neg\forall x\phi \vdash \exists x\neg\phi$ by (DM∧), and $\exists x\neg\phi \vdash \neg\phi$ by (Vac), proving the lemma. □

Lemma 13 *Suppose x doesn't occur free in ψ or any formula of T, and $T, \phi \vdash \psi$. Then $T, \exists x\phi \vdash \psi$.*

Proof By Ibn Sīnā's Rule, $T, \exists x\phi \vdash \exists x\psi$. Then by (Vac), $T, \exists x\phi \vdash \psi$. □

Lemma 14 *Suppose x doesn't occur free in any formula of T, and $T \vdash \phi$. Then $T \vdash \forall x\phi$.*

Proof First suppose T is nonempty; let ψ be any formula of T. Then $T, \psi \vdash \phi$, so by Ibn Sīnā's Rule $T, \forall x\psi \vdash \forall x\phi$. Hence $T, \psi \vdash \forall x\phi$ by Lemma 12, and so $T \vdash \forall x\phi$ since $\psi \in T$.
If T is empty then let ξ be any formula not containing x, and reason as above to get $(\xi \vee \neg\xi) \vdash \forall x\phi$. Then the result follows by (ExclM). □

We say that a set T of formulas is *inconsistent* if for some ζ, $T \vdash \zeta$ and $T \vdash \neg\zeta$. If T is not inconsistent we say it is *consistent*.

Lemma 15 *If T is inconsistent then for every formula ξ, $T \vdash \xi$.*

Proof If $T \vdash \zeta$ and $T \vdash \neg\zeta$, then $T \vdash (\zeta \wedge \neg\zeta)$ by (∧I). It follows by Lemma 10 that $T \vdash \xi$. □

Lemma 16 *If T is consistent and $T \vdash \chi$, then $T \cup \{\chi\}$ is consistent.*

Proof Suppose that $T \vdash \chi$ but $T \cup \{\chi\}$ is inconsistent. Then for some ζ, $T, \chi \vdash \zeta$ and $T, \chi \vdash \neg\zeta$. Then $T \vdash \zeta$ by Lemma 3, and $T \vdash \neg\zeta$ for the same reason; so T is inconsistent. □

Lemma 17 *(a) If T is a set of formulas and ϕ a formula such that $T \cup \{\phi\}$ is inconsistent, then $T \vdash \neg\phi$.*
(b) If T is a set of formulas and ϕ a formula such that $T \cup \{\neg\phi\}$ is inconsistent, then $T \vdash \phi$.

Proof (a) Assume $T \cup \{\phi\}$ is inconsistent. Then by (∧I), there is some ζ such that $T, \phi \vdash (\zeta \wedge \neg\zeta)$. So by the Deduction Theorem, Theorem 1(a), $T \vdash ((\zeta \wedge \neg\zeta) \vee \neg\phi)$, and then $T \vdash \neg\phi$ by Lemma 7.
(b) Similar, using Theorem 1(b). □

Theorem 2 (Completeness Theorem) *If $T \models \phi$, where we regard free variables as constants, then $T \vdash \phi$.*

Proof In fact we will prove that every consistent set of formulas has a model. To derive the theorem as stated, suppose we don't have $T \vdash \phi$. Then by Lemma 17(b), $T \cup \{\neg\phi\}$ is consistent, so it has a model. This model is a counterexample to $T \models \phi$, so the theorem follows.

Now changing notation, we will show that if T is any consistent set of formulas, then we can extend T to a Hintikka set without losing consistency; we allow ourselves to add new variables to the language along the way. It's a standard result that Hintikka sets have models. I won't define Hintikka set, because the claims below make clear what the requirements are.

Claim 1 *Suppose T is consistent and $(\phi \wedge \psi) \in T$. Then $T \cup \{\phi, \psi\}$ is consistent.*

Proof of claim By (∧EL), (∧ER) and Lemma 16. □ Claim

Claim 2 *Suppose T is consistent and $(\phi \vee \psi) \in T$. Then at least one of $T \cup \{\phi\}$ and $T \cup \{\psi\}$ is consistent.*

Proof of claim Suppose to the contrary that $T \cup \{\phi\}$ and $T \cup \{\psi\}$ are both inconsistent. Then by Lemma 17(a) we have $T \vdash \neg\phi$. But by Lemma 2 and the assumption that $(\phi \vee \psi) \in T$, we also have $T \vdash (\phi \vee \psi)$, so $T \vdash \psi$ by (ChrysL). Now the same argument as for ϕ shows that $T \vdash \neg\psi$, which establishes that T is inconsistent. □ Claim

Claim 3 *Suppose T is consistent and $\neg(\phi \wedge \psi) \in T$. Then at least one of $T \cup \{\neg\phi\}$ and $T \cup \{\neg\psi\}$ is consistent.*

Proof of claim By (DM∧) and Lemma 16, $T \cup \{(\neg\phi \vee \neg\psi)\}$ is consistent. So the claim follows by Claim 2. □ Claim

Claim 4 *Suppose T is consistent and $\neg(\phi \vee \psi) \in T$. Then $T \cup \{\neg\phi, \neg\psi\}$ is consistent.*

Proof of claim By (DM∨) and Lemma 16, $T \cup \{(\neg\phi \wedge \neg\psi)\}$ is consistent. So the claim follows by Claim 1. □ Claim

Claim 5 *Suppose T is consistent and $\neg\neg\phi \in T$. Then $T \cup \{\phi\}$ is consistent.*

Proof of claim This is by Lemma 4 and Lemma 16. □ Claim

Claim 6 *Suppose T is consistent and $\forall x\phi \in T$. Then $T \cup \{\phi[t/x] : t \text{ a variable}\}$ is consistent.*

Proof of claim By Lemma 1 it suffices to show that we can consistently add a finite number of $\phi[t/x]$ to T; and we can show this by induction, adding one formula at a time. So it suffices to show that if t is any variable, $T \cup \{\phi[t/x]\}$ is consistent. But we have this by (∀E) and Lemma 16. □ Claim

Claim 7 *Suppose T is consistent and $\exists x\phi \in T$, and let t be any variable not occurring in ϕ and not occurring free in any formula in T. Then $T \cup \{\phi[t/x]\}$ is consistent.*

Proof of claim Let ξ be any formula in which t doesn't occur free. If $T \cup \{\phi[t/x]\}$ is inconsistent then by Lemma 15, $T, \phi[t/x] \vdash (\xi \wedge \neg\xi)$. So by Lemma 13, $T, \exists t\phi[t/x] \vdash (\xi \wedge \neg\xi)$. But by (Var), $\exists x\phi \vdash \exists t\phi[t/x]$, which shows that T is already inconsistent. □ Claim

Claim 8 *Suppose T is consistent and $\neg\forall x\phi \in T$, and t has no free occurrence in $\forall x\phi$ or any formula in T. Then $T \cup \{\neg\phi[t/x]\}$ is consistent.*

Proof of claim By (DM∀) and Lemma 16, $T \cup \{\exists x\neg\phi\}$ is consistent. Then the claim follows using Claim 7. □ Claim

Claim 9 *Suppose T is consistent and $\neg\exists x\phi \in T$. Then $T \cup \{\neg\phi[t/x] : t \text{ a variable}\}$ is consistent.*

Proof of claim By (DM∃) and Lemma 16, $T \cup \{\forall x\neg\phi\}$ is consistent. Then the claim follows using Claim 6. □ Claim

Together the claims show that if T is consistent, it can be extended to a Hintikka set, in general in a language with more variables. With this the proof is complete. □

The calculus for $\mathcal{SL}$ certainly contains some redundancies. For example we never used (∃I). It can be derived from (∀E), but almost certainly Ibn Sīnā would have regarded it as obviously correct in itself. By the same token, perhaps we should have included the results of Lemmas 4 and 5 as axioms, since Ibn Sīnā would certainly have reckoned that they are more obvious in themselves than their proofs are.

2.4 Natural deduction

H14. Dag Prawitz, *Natural Deduction, A Proof-Theoretic Study*, Dover, New York 2006, p. 20.

$$\&\mathrm{I})\quad \frac{A \quad B}{A \,\&\, B} \qquad\qquad \&\mathrm{E})\quad \frac{A \,\&\, B}{A} \quad \frac{A \,\&\, B}{B}$$

$$\vee\mathrm{I})\quad \frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad\qquad \vee\mathrm{E})\quad \frac{A \vee B \quad \overset{(A)}{C} \quad \overset{(B)}{C}}{C}$$

$$\supset\mathrm{I})\quad \frac{\overset{(A)}{B}}{A \supset B} \qquad\qquad \supset\mathrm{E})\quad \frac{A \quad A \supset B}{B}$$

$$\forall\mathrm{I})\quad \frac{A}{\forall x A^{a}_{x}} \qquad\qquad \forall\mathrm{E})\quad \frac{\forall x A}{A^{x}_{t}}$$

$$\exists\mathrm{I})\quad \frac{A^{x}_{t}}{\exists x A} \qquad\qquad \exists\mathrm{E})\quad \frac{\exists x A \quad \overset{(A^{x}_{a})}{B}}{B}$$

$$\curlywedge_{\mathrm{I}})\quad \frac{\curlywedge}{A} \qquad\qquad \curlywedge_{c})\quad \frac{\overset{(\sim A)}{\curlywedge}}{A}$$