The influence of Augustus De Morgan

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De Morgan was a mathematician of very wide interests, more a publicist and educator than a researcher.

His specialist field was logic in the Aristotelian tradition. He played a major role in re-establishing an interest in logic, particularly among British and US mathematicians. But he had less originality and insight than some of his predecessors in this field, such as Avicenna (11th c), Buridan (14th c), Leibniz (17th c).

(1) Mathematical induction
(2) Quantifiers
(3) Relational logic
(4) The ‘De Morgan laws’
(5) De Morgan and present-day mathematics

(1) Mathematical Induction

$$\begin{aligned} 1 &= 1 = & 1^2 \\ 1+3 &= 4 = & 2^2 \\ 1+3+5 &= 9 = & 3^2 \\ 1+3+5+7 &= 16 = & 4^2 \end{aligned}$$

Etc. We observe:

Fact One. The equations can be put in a formula:

$$\text{For all } n \geqslant 1, \quad 1+3+\ldots+(2n-1) = n^2.$$

Write this as: For all $n \geqslant 1$, $\Phi(n)$.

Fact Two. For each n, we can prove $\Phi(n+1)$ from $\Phi(n)$:

$$\begin{aligned} 1 + \ldots + (2(n+1) - 1) &= 1 + \ldots + (2n-1) + (2n+1) \\ &= n^2 + (2n+1) \quad (\text{by } \Phi(n)) \\ &= (n+1)^2. \end{aligned}$$

Fact Three. So for each n, we can prove $\Phi(n)$ by proving first $\Phi(1)$, then proving $\Phi(2)$, then proving $\Phi(3)$, etc. up to proving $\Phi(n)$.

Fact Four (Mathematical Induction). Fact Three is unnecessary. All we need to do is to prove two things:

(1) $\Phi(1)$.
(2) For all n, if $\Phi(n)$ then $\Phi(n+1)$.

Then it follows (without any more work) that for all n, $\Phi(n)$.

Fact Four is now generally reckoned to have been first stated by Blaise Pascal in 1653.
De Morgan was apparently unaware of Fact Four, though he was clear about Facts One to Three.

De Morgan's paper on Mathematical Induction (1838) contains no statement of Fact Four.
I also checked his textbook *Elements of Arithmetic* (1846 edition), and found no statement of induction there either.

Both the paper and the book illustrate a feature of De Morgan's writing.
He preferred examples and analogies to precise statement.
This may be one reason why he never made it to the top level in mathematics.

C. S. Peirce's Obituary of De Morgan (1871) contains several 'remarks of a more critical nature'. Among them:

> 'His elementary books, which are not enough known, are excellent, especially for students who have no natural turn for mathematics.'
>
> 'It would be premature to try to say what the final judgment of De Morgan's [logical] system will be, but it may at least be confidently predicted that the logic of relatives, which he was the first to investigate extensively, will eventually be recognized as a part of logic.'

Contrary to what you may read, De Morgan did *not* invent the name 'mathematical induction'.
His name for Fact Three was 'successive induction'.

The phrase 'mathematical induction' seems to have come from the editors of the Penny Cyclopedia,
who in 1838 invited two authors to write,
one (De Morgan) about mathematical induction
and one (Hamilton?) about logical induction.

(2) Quantifiers

Aristotelian logic applies to sentences expressible in any of the four following forms:

(a) Every A is a B.
(e) No A is a B.
(i) Some A is a B.
(o) Not every A is a B.

A *syllogism* is an argument consisting of two sentences of these forms (the premises) and a third, also of these forms, which is derivable from them (the conclusion).
We require that three letters, say A, B, C, occur in the sentences, and each letter occurs once in each of two sentences. For example

Every B is a C.
Some A is a B.
Therefore some A is a C.

It was known from around 1300 that a syllogism is logically correct if and only if it meets two conditions:

(1) At most one premise is negative (i.e. of the form (e) or (o)), and if one premise is negative then the conclusion is also negative.
(2) The letter that occurs in both premises is distributed in at least one of them, and any letter that is distributed in the conclusion must be distributed also in a premise.

The *distributed* letters are A in (a), both A and B in (e), and B in (o).

Hamilton and De Morgan had different views on a popular topic of the time, how to make logic algorithmic.
(Cf. Euler, Jevons, Venn, Lewis Carroll, Peirce.)

De Morgan used the idea of distribution. He had a notation with parentheses to show which letters are distributed:

(A) is distributed
$)A($ is undistributed

Hamilton described De Morgan's parentheses as 'spiculae', a name that De Morgan accepted.

De Morgan allows us to drop one or both of the spiculae on a letter. So for example:

$(A))B($ (i.e. 'Every A is a B') shortens to $(A)B$

When Bertrand Russell in 1908 needed a notation for 'Every x', he wrote it

$$(x)$$

He gave no source, but it's hard to believe this is not just De Morgan's spicular notation.

Leaving off one more spicula gives

$A)B$ (I.e. 'If a thing is an A then it's a B')

So it's interesting to find that Peano in 1891 symbolised 'If p then q' as

$$p \;Ↄ\; q$$

expanding $)$ to Ↄ (which Russell changed to $\supset$).
Again Peano gave no credit, but it's hard to believe this is just a coincidence!

Hamilton took over from Jeremy Bentham (via Bentham's nephew George) the idea that we should think of the letters A, B, C as standing for sets, so that we can do simple set-theoretic calculations on them.
He described this process of passing to sets as 'quantification', as for example in

All horses = some quadrupeds.

De Morgan, in comments on Hamilton's 'quantification of the predicate' in 1862, referred to the expressions 'all' and 'some' as *quantifying words*, and he shortened this to *quantifiers*.
This is the source of the word 'quantifier'.

(3) Relational logic

Relations were first studied by Aristotle, mid 4th century BC.
For him a relation is a property that a thing has *in relation to something else.*

Thus a father is always a father of y, for some y.

Likewise son (of y), head (of y), knowledge (of y).

De Morgan suggested to make a mathematical theory of relations.
He proposed to symbolise them as functions, possibly many-valued.

If F is 'father of', then $F(geometer)$ means father of a geometer.
If A is 'ancestor of', then $A(geometer)$ means ancestor of a geometer.

He wrote $R))S$ for 'Every R of something is an S of that thing', so for example $F))A$.

He observed: $AA))A$, but not $FF))F$.

To express this he suggested saying that 'ancestor' is *transitive.*
This is the origin of the word in mathematics.

Very early on, Aristotelian logicians had studied the converses of relations.
E.g. 'child' is the converse of 'parent'.

De Morgan wrote R^{-1} for the converse of R.
He wrote r for 'Non-R'.

One of his theorems: If $LM))N$ then $L^{-1}n))m$.

This theory was developed much further by Peirce,
Ernst Schröder, and Alfred Tarski and his students.
Today it is a 'part of logic' (Peirce) but a very specialised one.

From today's perspective, the breakthrough in relational logic came when people stopped thinking of a relation as a kind of function, and treated it as a two-variable statement.

I.e. not 'father of y' but 'x is the father of y', $father(x, y)$.

This breakthrough was made by Peirce around 1880.
Then to compose two relations, we need to state a quantification. 'a is the head of the father of b' is

$$\text{There is } z : head(a, z) \text{ and } father(z, b).$$

Separating out quantification in this way, we quickly get the whole of first-order logic, as Peirce did in 1885.

More remarks of Peirce on De Morgan and logic of relations:

> 'Mr. De Morgan had made a good start with it ten years before, but I will say, without affectation, that I at once left his work far behind.'
>
> 'De Morgan was one of the best logicians that ever lived and unquestionably the father of the logic of relatives. Owing, however, to the imperfection of his theory of relatives, the new form, as he enunciated it, was a downright paralogism, one of the premises being omitted.'

The missing premise was that there are finitely many objects!

(4) The De Morgan laws

There are two De Morgan laws of propositional logic and two corresponding De Morgan laws of classes.

Propositional:

$$\begin{aligned} \text{not } (p \text{ or } q) &\equiv (\text{not-}p \text{ and not-}q) \\ \text{not } (p \text{ and } q) &\equiv (\text{not-}p \text{ or not-}q) \end{aligned}$$

Class form, writing $\overline{X}$ for the class of all things not in the class X:

$$\begin{aligned} \overline{(X \cup Y)} &= (\overline{X} \cap \overline{Y}) \\ \overline{(X \cap Y)} &= (\overline{X} \cup \overline{Y}) \end{aligned}$$

The propositional forms were common knowledge among Aristotelian logicians nearly a thousand years ago.
Avicenna in c. 1024 expressed one as follows:

> [This proposition] can be taken in two ways.
> One of them is that 'It is not greater [than Y] and not less [than Y]'.
> The second is 'It is not the case that it is either greater [than Y] or less [than Y]'.

De Morgan seems to have been the first person to state the class form, in 1858 and 1860:

> 'The contrary of an aggregate is the compound of the contraries of the aggregants:
> the contrary of a compound is the aggregate of the contraries of the components.'

NB He says it in English, not in symbols.
This is because his notation for negation or complement applies only to single letters,
e.g. the complement of R is r (as in relation theory).
So De Morgan's notation couldn't state De Morgan's laws!

Peano picked up the class laws in 1897 because he could use them as axioms.
In 1910 Whitehead and Russell included the propositional form of the laws in *Principia Mathematica*.

The propositional form was called 'De Morgan's theorem' by C. I. Lewis in 1912, and this name is found still in Claude Shannon's Master's Thesis on switching circuits in 1936.
The change to 'De Morgan's law(s)' was made by Tarski and his student J. C. C. McKinsey in 1940.

(5) De Morgan's mathematical influence?

Hard to say. It's much easier to ask:

> If De Morgan is looking down on us from his Spiritualist heaven,
> what in modern mathematics will most please him?

Obviously things like the new LMS Shephard Prize for 'a contribution to mathematics with a strong intuitive component which can be explained to those with little or no knowledge of university mathematics'.

Also various Chairs related to the public understanding of mathematics.

Also he would surely be pleased to see how the theory of relations developed.

Whitehead and Russell made it a part of logic in *Principia Mathematica* (1910).

Tarski and his students and colleagues (including Lyndon) turned it into a subtle and sophisticated theory of relation algebras.

Perhaps best of all, two London mathematicians (Robin Hirsch and Ian Hodkinson) recently re-worked Tarski's theory of relation algebras,
using the mathematical theory of games as a tool for both research and exposition.
There are several reasons why De Morgan would have liked this use of games.

Robin Hirsch is a mathematical professor in the Computer Science Department at University College London,
so he is a direct successor of De Morgan.