In pursuit of a medieval model theory

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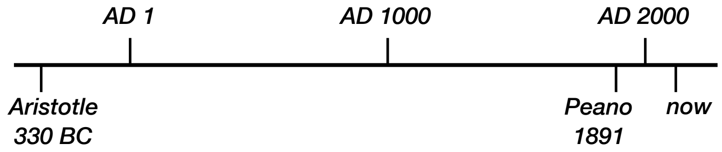
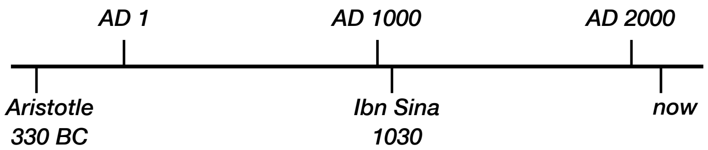
This meeting invites us to take a perspective on logic as a whole. Historically, here is Western logic as a whole:



All logicians traced their work back to Aristotle,
until Peano in 1891 introduced a new discipline
'Mathematical logic', whose signs allow us to represent
'all the propositions of algebra and geometry'.

Why did the escape from Aristotle come so late?
By 1891, astronomers and medics had long since abandoned Aristotle's assumptions.

Here is Middle Eastern logic for comparison:



The abandonment of Aristotle occurred around 1030, nearly a thousand years ago, with the *Ishārāt* of Ibn Sīnā (= Avicenna).

Why was the escape so much earlier in the East?
Why did it still take over a thousand years to move on from Aristotle?
What did the escape consist of?

These questions are too large and abstract to allow any informative answer.
So I restrict to a more specific issue that seems to be both central and connected with the escape (both East and West).
Namely, how the notion of logical consequence was used.
(NB how it was *used*, not how it was *justified*, which is an issue in philosophy of logic rather than logic itself.)

'θ is a logical consequence of premises Φ'

means

(Proof-theoretically) *There is* a pattern of inferences (i.e. a derivation) that leads from Φ to θ.

(Model-theoretically) *Every* interpretation that makes Φ true (i.e. every model of Φ) also makes θ true.

The difference between 'There is' and 'Every' implies that these two notions of logical consequence will be applied in very different ways.

For example, showing that θ is a logical consequence of Φ proof-theoretically just needs one derivation.
But showing it model-theoretically needs some kind of survey of *all* models of Φ—there could be infinitely many.

In the West: In propositional logic, use of truth values allows model-theoretic proofs by truth-tables. This dates back to Wittgenstein and Post around 1920, and Peirce a little earlier.

In 1936 Tarski wrote a paper defining model-theoretic consequence.
More significant is papers of 1949–1954 in which Tarski and Abraham Robinson introduced model-theoretic consequence as a key concept of model theory.

In the East: Abū al-Barakāt in the mid 12th century showed how to prove categorical syllogisms model-theoretically.

Already in the 1020s Ibn Sīnā started to develop a model-theoretic approach for showing that θ *is not* a logical consequence of Φ.

(We will study both these developments below.)

So in both the East and the West, use of model-theoretic consequence appears around or soon after the escape from Aristotle.

Could it be that something in Aristotle's logic prevented logicians from taking up model-theoretic consequence?

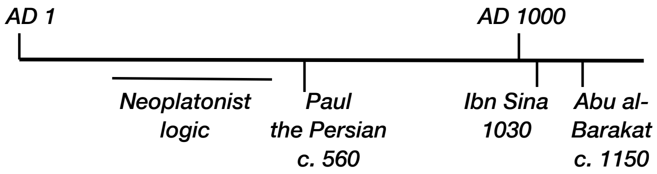
Two facts against this:

(i) We can trace the origins of Abū al-Barakāt's innovation, and it is clear that they go back to Aristotle himself.
They were natural and intrinsic developments of ideas already used by Aristotle.

(ii) In the East the model-theoretic idea never took hold, and the work of Ibn Sīnā and Abū al-Barakāt was either misunderstood or ignored. Absence of Aristotle was not a sufficient condition for healthy development.

We will sketch the development from Aristotle to Abū al-Barakāt, and then draw tentative conclusions.

- Aristotle's categorical syllogisms
- Abū al-Barakāt's proofs
- Anticipations in Paul the Persian and Ibn Sīnā



Aristotle had two logics: categorical and modal.
A typical categorical syllogism:

Every A is a B. No B is a C. Therefore no A is a C.

A typical modal syllogism:

Every A can be a B. No B has to be a C. Therefore no A has to be a C.

The logic that held Western thinking for over 2000 years was the categorical. Here we will ignore the modal.

In categorical logic we take a pair Φ of formal sentences, and a set $\theta_1, \ldots, \theta_4$ of four possible logical consequences; we determine *either* that some θ_i is the strongest θ_i that is a logical consequence of Φ,
or that none of the θ_i is a logical consequence of Φ.
In the first case we say that Φ is 'productive' with the given 'conclusion', in the second case it is 'nonproductive'.

Aristotle gave a proof-theoretic method for proving productivity and deriving the conclusion,
and a model-theoretic method for proving nonproductivity.

Example of derivation, proving that 'Some A is a C' is a logical consequence of 'Some B is a C, and every B is an A'.

$$\dfrac{\dfrac{\dfrac{\text{Some } B \text{ is a } C.}{\text{Some } C \text{ is a } B.} \qquad \text{Every } B \text{ is an } A.}{\text{Some } C \text{ is an } A.}}{\text{Some } A \text{ is a } C.}$$

Example of proof that ‘Every A is a B, and some B is not a C’ is nonproductive:

(1)	(2)
All wisdom is a state.	All ignorance is a state.
Some states are not good.	Some states are not good.
(Pseudo)conclusion: All wisdom is good.	*(Pseudo)conclusion:* No ignorance is good.

Aristotle infers (correctly, given how he understands the sentence forms) that none of the four possible conclusion forms can be a logical consequence of the premises.
Aristotle himself calls the pseudoconclusions just ‘conclusions’. They are true in the interpretations, not derived from the premises.

Remark One: Aristotle’s proof of nonproductivity gives two models of the premises, one of which is a model of ‘Every A is a C’ and the other a model of ‘No A is a C’.
He is not necessarily using model-theoretic consequence; he may just be assuming that his inference rules never deduce a falsehood from truths.

Remark Two: To operate Aristotle’s nonproductivity method, we need to know which bits of sentences we can replace in giving the interpretations. So Aristotle’s method works with argument *forms* (based on sentence forms), and hence it proves *formal* non-consequence.

We move on to Abū al-Barakāt bin Malka al-Baghdādī, now believed to be the same person as the Talmudic scholar Rabbi Baruch ben Melekh.
He lived from c. 1085 to c. 1170. He was a Baghdad Jew who converted to Islam late in life, probably under duress.

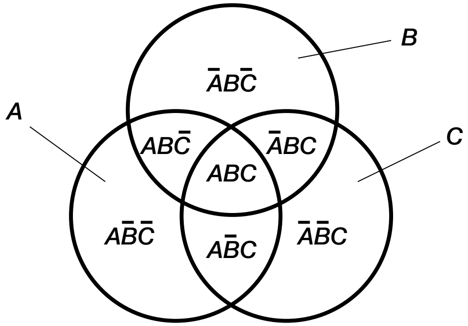
In the West he is best known as the first person to state that bodies fall with constant acceleration.

He claimed to have read Aristotle’s logic, but his many differences from Aristotle suggest he hadn’t.
He did read Ibn Sīnā in detail.

Abū al-Barakāt explains categorical logic by listing 48 pairs of premises.
For each productive pair he gives between two and four interpretations.
For each nonproductive pair he gives three interpretations.

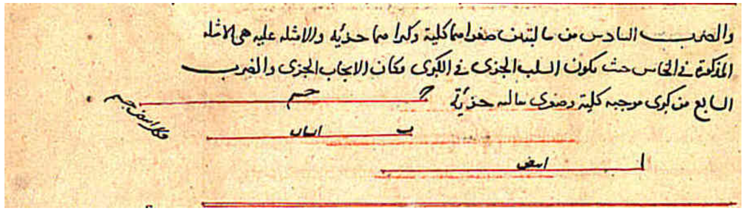
The nonproductive case is as Aristotle, except that Abū al-Barakāt adds an unnecessary third interpretation satisfying ‘Some but not every A is a C’.
(Evidence that Abū al-Barakāt hadn’t read Aristotle!)

In the productive case, Abū al-Barakāt argues model-theoretically.
There will be infinitely many interpretations that are models of the premises.
But what sentences are true in an interpretation depends only on which bounded areas in the diagram are nonempty:



So we need consider at most $2^7 = 124$ interpretations.
Restricting to models of the premises takes the number required to 16 in the worst case.

Abū al-Barakāt doesn't have Venn diagrams.
Instead he invents his own notation:



(The lines are for 'bodily', 'human', 'white'.)

For example to prove

> Every A is a B. Every B is a C.
> Therefore every A is a B.

Abū al-Barakāt gives four interpretations as follows:

These represent the four relevantly different models of the premises.
In each case 'Every A is a C', so this is a model-theoretic logical consequence of the premises.

Abū al-Barakāt never gives more than four interpretations, though some later cases need up to sixteen.
Probably he thinks he gives enough to show the method, and his readers won't thank him for running through sixteen cases!

In any case, later in the list of premise-pairs he sometimes fails to give any proof at all. But the examples that he does give in full are certainly enough to show the method.

Remark on Abū al-Barakāt's diagrams. Until last year, the earliest examples known to Western logicians of diagrams for proving logical consequences were those of Leibniz.
Abū al-Barakāt's take them back a further 500 years.

Abū al-Barakāt's diagrams represent *interpretations*, like those of Gergonne (1816). By contrast the diagrams of Leibniz, Euler, Venn and Carroll represent *propositions.*
Curiously al-Ṭūsī, who reported Abū al-Barakāt's diagrams in his textbook of around 1230, got them wrong and thought they represented propositions.

Abū al-Barakāt was apparently the first person to use model-theoretic consequence in order to prove logical consequences (and the last person for some 700 years).
But about a hundred years earlier, Ibn Sīnā started to develop model-theoretic consequence for showing *logical non-consequences.*

This is in the radical section vi.2 of his book *Qiyās.*
In modern terms, he axiomatises the theory of power-set algebras in a fragment of first-order logic with primitive symbols $\subseteq$ and complement $^-$.

Up to choice of variable, there are eight atomic or negated atomic formulas:

$$A \subseteq C, \quad A \subseteq \overline{C}, \quad \overline{A} \subseteq C, \quad \overline{A} \subseteq \overline{C},$$

$$A \not\subseteq C, \quad A \not\subseteq \overline{C}, \quad \overline{A} \not\subseteq C, \quad \overline{A} \not\subseteq \overline{C}$$

Ibn Sīnā gives proofs of nonproductivity of pairs ϕ, ψ of atomic or negated atomic formulas.

Aristotle's method using two pseudoconclusions won't work; there are too few logical relations between atomic formulas.

Nevertheless for each nonproductive pair ϕ, ψ Ibn Sīnā gives just two interpretations.
His idea seems to be that for each possible conclusion θ, one of the two interpretations falsifies θ.

He gives 16 examples. Some of them don't work at all.
But most do, *if we give the interpretations suitably restricted universes.*
This seems to be the first appearance of 'models' in the modern sense, with a universe and relations defined on the universe. But more study of Ibn Sīnā's text is needed.
I believe later Arabic logicians ignored this work of Ibn Sīnā.

Various conclusions

Aristotle's categorical logic contained ideas that could be developed into model-theoretic consequence.
But it took over 1300 years for this development to happen; and when it did, it was quickly forgotten.

When these ideas resurfaced in the West in around 1900, the impetus came from *geometry*, not from inside logic.
Cf. Hilbert *Grundlagen der Geometrie* 1899, and related earlier work of Peano 1894, stimulated by the Klein-Beltrami model of hyperbolic space.

Why the slow start?

Until around the time of Cicero (1st century BC), logic was known only to a few specialists.
When logic became public knowledge, Aristotelian logic was already absorbing or crowding out its rivals, e.g. Stoic logic.

Quite early in the Roman empire, Aristotelian logic was adopted by the leading pagan philosophers, the Neoplatonists, as a first step in their education programme.
It was purely a tool for teaching discipline of thought.
Neoplatonists had no interest in developing it for its own sake.
It was not generally understood as *formal* logic—rather as a system of heuristics and good style.

With the collapse of the pagan schools, under pressure from the Christians and then the Islamic empire, specialists took themselves and their books to the Middle East.

Thus Paul the Persian (c. 560) wrote an introduction to logic which survives in Syriac. He or his sources were interested in the structure of logic for its own sake. He presents the difference between productive and nonproductive premise pairs as a difference between the *families of interpretations which make them true.* This looks like a pointer to what Abū al-Barakāt did later. (But I think I may be the only modern logician to have looked at Paul the Persian. Work to do.)

In the mainstream of Islamic culture, logic was sometimes adopted as an educational tool.
It still figures in the syllabus of the Iranian madrasas.
Abū al-Barakāt's idea could have been useful here, like Venn diagrams in the West.
But apparently there was nobody with an interest in education who understood Abū al-Barakāt well enough.

It seems that new logical ideas thrive only when there are enough people with the motivation to pursue them.
The motivation need not come from within logic.
(And this is one reason why the future development of logic is hard to predict.)

There is another reason why Abū al-Barakāt's ideas didn't catch on.
The natural next step would have been to apply model-theoretic consequence to another logic studied by Ibn Sīnā. This was his 'two-dimensional' temporal logic, which he developed as a semantics for rejigging Aristotle's modal logic.

Just as with categorical logic, we can show that for each interpretation of three relations, there are finitely many yes/no facts about the interpretation that determine which of Ibn Sīnā's temporal sentences are true in it.
But back-of-envelope calculations suggest that we would need to consider not 2^7 but $2^{2^8} = 2^{256} \sim 10^{77}$ interpretations.

This implies that Abū al-Barakāt was near the limit of what could be done with model-theoretic consequence using the then-standard method of examining many individual cases (*istikhrāj*).

Progress would have needed a set of concepts for *abstract* development of the subject, in other words a *model theory*. Frege complained in 1906 that there was still no such theory adequate to cover non-entailment.

"How can one prove the independence of a thought from a group of thoughts? . . . [W]ith this question we enter a realm that is otherwise foreign to mathematics. . . . Now we may assume that this realm has its own specific, basic truths which are as essential to the proofs constructed in it as the axioms of geometry are to the proofs of geometry; and that we also need these basic truths especially to prove the independence of a thought from a group of thoughts. . . . But we are here in unexplored territory."
(Frege, *Über die Grundlagen der Geometrie III*)

And again, this 'new realm' is not likely to be developed until problems have arisen that motivate people to develop it.

Wilfrid Hodges, 'Two early Arabic applications of model-theoretic consequence', *Logica Universalis* online (2018) https://doi.org/10.1007/s11787-018-0187-6.

Wilfrid Hodges, 'Identifying Ibn Sīnā's hypothetical logic: II. Interpretations', draft online at http://wilfridhodges.co.uk/arabic59a.pdf.

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