ON ω_1-CATEGORICAL BUT NOT ω-CATEGORICAL THEORIES

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by
William Marsh

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ABSTRACT

By generalizing to model theory notions connected with dimension, it is shown that certain theories categorical in uncountable powers have countably many denumerable isomorphism types which are arranged in an $\omega + 1$ sequence under the ordering of the possibility of elementary imbedding. It is also shown that any countable elementary extension of a denumerable saturated model of a theory categorical in uncountable powers is saturated.

PREFACE

This thesis should be readable by anyone with a little background in logic; an excellent place to get such a background is the expository paper [7] of R. L. Vaught.

Professor Vaught pointed out an error in a preliminary version of this paper and made several useful suggestions about how to proceed. Professor M. Morley in letters and conversation was most helpful and, in particular, pointed out to me a theorem which is implicit in [3] that turned out to be the key to applying the results of Chapter 1 of this thesis to theories categorical in ω_1 but not ω. I would like to take this opportunity to thank both of these men.

I would like to thank my advisor, Professor Donald Kreider, who was more than generous with his time and help.

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Introduction

The Skolem-Löwenheim Theorem says that if a first order theory in a denumerable language has an infinite model, it has at least one of every infinite cardinality. Łoś in [1] gave examples of theories T_1, T_2, and T_3 such that T_1 was categorical for all infinite cardinals, T_2 was categorical for all uncountable cardinals but not ω, and T_3 was categorical in ω, but for no other infinite cardinals. Morley in [3] showed that if a theory is categorical in one uncountable cardinal, it is categorical in all of them. Morley gave in [2] a characterization of theories categorical in uncountable powers and used it to prove (oral communication) that there are at most countably many denumerable models of such a theory. The main results in this thesis are that a certain class of ω_1- but not ω-categorical theories have infinitely many non-isomorphic countable models and that an elementary extension of a countable saturated model of an ω_1-categorical theory is saturated.

Chapter 0 is notation and standard definitions. Chapter 1 consists of generalizing certain algebraic concepts to a class of first-order theories and concludes with a generalization of the Steinitz Theorem. In Chapter 2 the methods of the preceding chapter are combined with known results to prove the results mentioned in the previous paragraph.

Vaught pointed out to the author that the notion of algebraic closure in Chapter 1 is identical to that of obligation in [5], where Park uses it to investigate intersection properties of models. This notion and that of strongly minimal set are closely related to Morley's algebraic points and points transcendental in rank one, respectively (in [3]). The notion of strongly minimal set was known to Vaught. He, and probably others, knew the Steinitz Theorem could be generalized, though the exact form in Chapter 1 is perhaps new. To the author's best knowledge the formulation of the concept of dimension presented here is also new.

Chapter 0

An ordinal is identified with the set of its predecessors. $\omega = \omega_0$ is the first infinite ordinal and ω_1 the first uncountable ordinal. If X is any set, card(X) is the first ordinal which can be put in one-one correspondence with X. An enumeration x_i of X is a one-one function from card(X) onto X.

A similarity type τ is a function from an ordinal λ into ω.

A structure $A = \langle |A|, R_i^A \rangle_{i<\lambda}$ of similarity type $\tau \in \omega^\lambda$ is a set $|A|$ called the domain of A and a $\tau(i)$-ary relation R_i^A if $\tau(i) > 0$, $i < \lambda$, and a distinguished element $a_i = R_i^A \in |A|$ if $\tau(i) = 0$, $i < \lambda$.

L_τ for $\tau \in \omega^\lambda$ is the set of formulas of the first order language with equality which has a $\tau(i)$-ary predicate symbol R_i for each $\tau(i) > 0$, $i < \lambda$, and an individual constant symbol $c_i = R_i$ for each $\tau(i) = 0$, $i < \lambda$. S_τ is the set of sentences of L_τ and F_τ^j for $j < \omega$ is the set of formulas whose free variables are among $\{v_0, v_1, \ldots, v_{j-1}\}$ where v_0, v_1, $\ldots$ are the variables used in L_τ ($S_\tau = F_\tau^0$). A complete theory T in L_τ is a subset of S_τ such that $T \vdash \phi$ implies $\phi \in T$ and for every $\phi \in S_\tau$, either $\phi \in T$ or $\neg\phi \in T$, but not both.

If T is a complete theory in L_τ then $\phi \sim \psi$ iff $T \vdash \forall v_0 \ldots \forall v_{n-1}[\phi \leftrightarrow \psi]$ is an equivalence relation on F_τ^n. The equivalence classes $[\phi]$ under $\sim$ along with the operations induced on them by $\wedge$, $\vee$, and $\neg$ are a Boolean algebra denoted by $B^n(T)$. $P^n(T)$ is the set of ultrafilters in $B^n(T)$. $L(T)$ is L_τ.

If A is of similarity type τ and $\phi \in F_\tau^n - F_\tau^{n-1}$ for $n > 1$, then ϕ^A is the n-ary relation on $|A|$ determined by $\langle a_0, \ldots, a_{n-1}\rangle \in \phi^A$ iff $\langle a_0, \ldots, a_{n-1}\rangle$ satisfies $\phi(v_0, \ldots, v_{n-1})$ in A; note that $\phi \sim \psi$ iff $\phi^A = \psi^A$ for models A of T. For $\phi \in F_\tau^n - F_\tau^{n-1}$, $\phi^A(v_0, \ldots, v_{k-1}, a_k, \ldots, a_{n-1})$ is the set of all $\langle a_0, \ldots, a_{k-1}\rangle$ such that $\langle a_0, \ldots, a_{k-1}, a_k, \ldots, a_{n-1}\rangle \in \phi^A$. $\{[\phi] \in B^n(T) | \langle a_0, \ldots, a_{n-1}\rangle \in \phi^A\}$ is an ultrafilter in $B^n(T)$ and is said to be realized by $\langle a_0, \ldots, a_{n-1}\rangle$ in A. th(A) is the set of all $\phi \in S_\tau$ which are true in A; $L(A)$ is $L(\text{th}(A))$.

If A is a structure of type $\tau \in \omega^\lambda$ and $\mu < \lambda$, then the structure $\langle |A|, R_i\rangle_{i<\mu}$ is said to be the μ-reduct of A and A is an expansion of $\langle |A|, R_i\rangle_{i<\mu}$. If $X \subseteq |A|$ and x_i for $i <$ card(X) is an enumeration of X, then (A, x_i) is the structure $\langle |A|, S_i^A\rangle$ of type $\sigma \in \omega^{\lambda+\text{card}(X)}$ where

$$\sigma(i) = \begin{cases} \tau(i) & \text{if } i < \lambda \\ 0 & \text{if } \lambda \leqslant i < \lambda + \text{card}(X) \end{cases}$$

and

$$S_i^A = \begin{cases} R_i^A & \text{if } i < \lambda \\ x_j & \text{if } i = \lambda + j < \lambda + \text{card}(X). \end{cases}$$

Notice that th$(A) \subseteq$ th$((A, x_i))$. If $X = \{a_0, \dots, a_k\}$ and $x_i = a_i$ $0 \leqslant i \leqslant k$ then th$((A, x_i))$ consists of all formulas obtained by substituting $c_{\lambda+i}$ for v_i (when free) in the formulas ϕ such that $[\phi]$ is in the ultrafilter realized by $\langle a_0, \dots, a_k\rangle$. Thus the elements of $P^n(T)$ correspond to consistent complete extensions of T by adding n new individual constants to the language L_τ. We will generally call L_σ the language appropriate to (A, x_i) and will suppress the σ and the enumeration; if $x_i = a_i$, $0 \leqslant i \leqslant k < \omega$ we would write (A, x_i) as $(A, a_0, \dots, a_k)$.

If A and B are structures then A is elementary equivalent to B, $A \equiv B$, if th$A) =$ th(B). If $X \subseteq |A|$ and $f : X \to |B|$ then f is an elementary monomorphism if $(A, x_i) \equiv (B, f(x_i))$ for any enumeration x_i of X. A is an elementary substructure, $A \prec B$, of B if $|A| \subseteq |B|$ and the identity map of $|A|$ into $|B|$ is an elementary monomorphism. An isomorphism of A onto B is an elementary monomorphism of $|A|$ onto $|B|$. A theory T is λ-categorical if it has a model A with card$(|A|) = \lambda$ and every two such are isomorphic.

We will use $\exists^k!v\phi(v)$ to mean there exist exactly k v such that $\phi(v)$, for any $k < \omega$; $\exists^k!x\phi(x)$ is an abbreviation for the formula

$$\exists v_{j_1} \exists v_{j_2} \dots \exists v_{j_k} \left[\bigwedge_{i=1}^{k} \phi(v_{j_i}) \wedge \forall v_{j_{k+1}} \left(\phi(v_{j_{k+1}}) \to \bigvee_{i=1}^{k} v_{j_{k+1}} = v_{j_i} \right) \right],$$

where the variables v_{j_i} are chosen to avoid clashes.

Chapter I

Throughout this chapter T is a complete theory in L_τ and L_τ is countable, i.e., $\tau \in \omega^\lambda$ with $\lambda < \omega_1$. A, B, and C are always models of T.

The definitions in this chapter are motivated and the theorems suggested by the two following examples of theories which are ω_1- but not ω-categorical.

Example 1. The theory of algebraically closed fields of characteristic 0. The denumerable models of this theory are algebraically closed fields whose degree of transcendence over the rationals is countable.

Example 2. The theory of torsion-free abelian quotient groups. These are precisely the vector spaces over the rationals considered as groups, and the denumerable models are those whose dimension i is such that $0 < i \leqslant \omega$.

Definition 1. Let $X \subseteq |A|$. Then the *algebraic closure* $\mathrm{cl}(X)$ of X is the union of all finite subsets of $|A|$ definable in (A, x_i) for x_i an enumeration of X. I.e., $\mathrm{cl}(X) = \bigcup\{\psi^{A'} | A' = (A, x_i), T' = \mathrm{th}(A'), [\psi] \in B^1(T'), \psi^{A'}$ is finite$\}$. We say X *spans* Y if $\mathrm{cl}(X) = Y$.

In Example 1 $\mathrm{cl}(\emptyset) =$ the algebraic numbers. In Example 2 $\mathrm{cl}(\emptyset) =$ the trivial subgroup. We note that $\mathrm{cl}(X)$ does not depend on the particular enumeration of X used. Since L_τ is denumerable $\mathrm{card}(\mathrm{cl}(X)) \leqslant \max(\mathrm{card}(X), \omega)$. Also if $A \prec B$, $\mathrm{cl}(X)$ in B is the same as $\mathrm{cl}(X)$ in A, since if $\phi^B\langle v_0, x_1, \dots, x_n\rangle$ for $x_1, \dots, x_n \in X$ is finite of cardinality k, then $T' \vdash \exists^k ! v\phi(c_{\lambda+i_1}, \dots, c_{\lambda+i_n})$, so $\phi_A(v_0, x_1, \dots, x_n)$ must contain exactly k elements also, and by the definition of $A \prec B$, these must be the same. Finally, $x \in \mathrm{cl}(X)$ implies $x \in \mathrm{cl}(X_0)$ for some finite $X_0 \subseteq X$.

Proposition 1. For any subsets X and Y of $|A|$,

(i) $X \subseteq \mathrm{cl}(X)$

(ii) if $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

(iii) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

Proof:

(i) $v_0 = c_{\lambda+i}$ defines the unit set $\{x_i\}$.

(ii) The language for (A, y_i) contains formulas equivalent to any in the language for (A, x_i).

(iii) $\mathrm{cl}(X) \subseteq \mathrm{cl}(\mathrm{cl}(X))$ by (i). To prove the inclusion the other way assume $a \in \mathrm{cl}(\mathrm{cl}(X))$. I.e., assume that $a_1, \dots, a_n \in \mathrm{cl}(X)$, that $\phi^A(a, a_1, \dots, a_n)$, and that there are exactly k elements $y \in |A|$ such that $\phi^A(y, a_1, \dots, a_n)$. Since each

$a_j \in \mathrm{cl}(X)$, a_j is in a finite subset of A defined by some formula in $L(\mathrm{th}(A, x_i))$. Let ψ_j^A be the smallest such set, for each j, $0 \leqslant j \leqslant n$. Then $[\psi_j]$ is an atom in $B^1(\mathrm{th}(A, x_i))$ for each j, $0 \leqslant j \leqslant n$. Also $\mathrm{th}(A, x_i) \vdash \exists^{m_j}!v\psi_j(v)$ for some m_j for each j, $0 \leqslant j \leqslant n$. Since a_n satisfies $\exists^k!v\phi(v, a_1, \dots, a_n)$, so does every element of $\psi_n^A(v_0)$. But then there are at most $m_n \cdot k$ elements of $|A|$ in $\phi_0^A(v_0, a_1, \dots, a_{n-1})$ where $\phi_0(v_0, v_1, \dots, v_{n-1})$ is the formula $\exists v_n[\psi_n(v_n) \wedge \phi(v_0, \dots, v_n)]$. Continuing in this way we successively eliminate all of the a_i's and we have that $a \in \mathrm{cl}(X)$.

In terms of the algebraic closure of a set we can introduce other notions from algebra, in particular, that of an algebraIcally independent set.

Definition 2. $X \subseteq |A|$ is independent if for all $x \in X$, $x \notin \mathrm{cl}(X - \{x\})$.

It follows from Proposition 1 (ii) that any subset of an independent set is independent. Also any independent set is disjoint from $\mathrm{cl}(\emptyset)$.

Definition 3. A set ϕ^A (and the formula ϕ) is called *minimal* if ϕ^A is infinite and $\phi^A \cap \psi^A$ is either finite or cofinite in ϕ^A for every $\psi \in L(A)$. ϕ^A (and ϕ) is called *strongly minimal* if ϕ^A is minimal in any consistent extension of $\mathrm{th}(A)$ by constants. Let $M(A)$ denote the union of all strongly minimal subsets of $|A|$.

We note in passing that if $M(A) \neq \emptyset$, $\mathrm{cl}(\emptyset) \subseteq M(A)$ since P strongly minimal implies $P \cup \psi^A$ is strongly minimal for any finite $\psi^A \subseteq |A|$.

Before proving the next proposition we wish to point out a consequence of the Compactness Theorem: If $y \in |A| - \mathrm{cl}(\emptyset)$, then there is a structure A' such that $A \prec A'$ and there are infinitely many elements $b_i \in |A'|$, $i < \omega$, such that each b_i realizes the ultrafilter in $B^1(T)$ realized by y.

Proposition 2. Let $X \subseteq |A|$, $x \in M(A)$, $x \notin \mathrm{cl}(X)$ and $y \in \mathrm{cl}(X \cup \{x\}) - \mathrm{cl}(X)$. Then $x \notin \mathrm{cl}(X \cup \{y\})$.

Proof: Making use of the previous comment we can replace A by A', if necessary, so we will assume that the ultrafilter realized by y is realized by infinitely many elements of $|A|$. Also, since $\mathrm{cl}(X)$ in A is the same as $\mathrm{cl}(\emptyset)$ in (A, x_i), it is sufficient to prove the proposition for the case $X = \emptyset$.

Since $x \in M(A)$, $x \in \phi_1^A$ for some strongly minimal ϕ_1. Since $y \in \mathrm{cl}(\{x\})$, there is a $\phi_2 \in F_\tau^2$ such that $\phi_2^A(y, x)$ and there are exactly k a's in $|A|$ such that $\phi_2^A(a, x)$ for some $k < \omega$. Let $\phi_0(v_0)$ be the formula $\phi_1(v_0) \wedge (\exists^k!v_1)\phi_2(v_1, v_0)$. Then $x \in \phi_0^A \subseteq \phi_1^A$ so that ϕ_0^A is cofinite in ϕ_1^A, since $x \notin \mathrm{cl}(\emptyset)$. Note that ϕ_0^A is strongly minimal. Let $\psi(v_0)$ be the formula $\exists v_1[\phi_0(v_1) \wedge \phi_2(v_0, v_1)]$. Since $y \in \psi^A$, ψ^A is infinite. Let $\psi_i(v_0)$ be, for $0 < i < \omega$, the formula $(\exists^i!v_1)[\phi_0(v_1) \wedge \phi_2(v_0, v_1)]$. We claim that $y \in \psi_j^A$ for some j. If not, $y \in A_\omega = \psi^A - \bigcup_{0<i<\omega} \psi_i^A$. Let $y = a_0$ and $a_0, a_1, \dots, a_k$ be elements which realize the same ultrafilter as y; then $a_0, a_1, \dots, a_k \in A_\omega$. Let $P_i = \{a | \phi_0^A(a) \wedge \phi_2(a, a_i)\}$ for $0 \leqslant i \leqslant k$. Then by the construction of A_ω, each P_i is an infinite subset of ϕ_0^A. But each element of ϕ_0^A is related to exactly k elements by ϕ_2^A, so $\bigcap_{i=0}^k P_i = \emptyset$. Therefore for some i, $\phi_0^A - P_i$ is infinite, which contradicts the strong minimality of ϕ_0^A. Thus $y \in \psi_j^A$, $j < \omega$. But then $x \in \mathrm{cl}(\{y\})$.

Theorem 1. Let $X \subseteq M(A)$, $Y \subseteq M(A)$, $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$ and X be independent. Then (i) $\mathrm{card}(X) \leqslant \mathrm{card}(Y)$ and (ii) there is a $Y_0 \subseteq Y$ such that $\mathrm{cl}(X \cup Y_0) = \mathrm{cl}(Y)$ and $X \cup Y_0$ is independent.

Proof: (i) Let $X = \{x_i\}$, $i < \lambda = \mathrm{card}(X)$, and $Y = \{y_i\}$, $i < \mu = \mathrm{card}(Y)$. We will define y_i' and $X_i = \{x_j | i \leqslant j < \lambda\} \cup \{y_j' | j < i\}$, such that $\mathrm{cl}(X_i) = \mathrm{cl}(X)$ for all $i \leqslant \lambda$. Note that $X_0 = X$ and $X_\lambda \subseteq Y$, so that $\mathrm{cl}(X_0) = \mathrm{cl}(X)$ and X_0 is independent.

Assume $\mathrm{cl}(X_i) = \mathrm{cl}(X)$ and X_i is independent. There is some $y \in Y - \mathrm{cl}(X_i - \{x_i\})$, otherwise $Y \subseteq \mathrm{cl}(X_i - \{x_i\})$ and therefore $x_i \in \mathrm{cl}(Y) \subseteq \mathrm{cl}(X_i - \{x_i\})$ contradicting the independence of X_i. Let y_i' be the first such y in the enumeration y_j of Y. Then $x_i \in \mathrm{cl}((X_i - \{x_i\}) \cup \{y_i\}) = \mathrm{cl}(X_{i+1})$ by Proposition 2. Therefore, $\mathrm{cl}(X_{i+1}) = \mathrm{cl}(X_i) = \mathrm{cl}(X)$. $y_i' \in \mathrm{cl}(X_{i+1} - \{y_i\})$ by construction of X_{i+1}. Suppose $x \in \mathrm{cl}(X_{i+1} - \{x\})$ for some other $x \in X_{i+1}$. Then $x \notin \mathrm{cl}(X_{i+1} - \{x, y_i'\})$, since $X_{i+1} - \{y_i'\} \subseteq X_i$, which is independent; and subsets of independent sets are independent. Since $y_i' \in M(A)$ we can apply Proposition 2 again and get $y_i' \in \mathrm{cl}(X_{i+1} - \{y_i\})$ which we just saw can not be. We have proved that X_{i+1} is independent and $\mathrm{cl}(X_{i+1}) = \mathrm{cl}(X)$. Obviously, X_δ is independent and $\mathrm{cl}(X_\delta) = \mathrm{cl}(X)$ for any limit ordinal δ if the same is true for all its predecessors. The y_i' are distinct, so (i) is proved.

(ii) Let $Y = \{y_i\}$, $i < \mu = \mathrm{card}(Y)$. Define $Y_0 = \{y_i | y_i \notin \mathrm{cl}(X \cup \{y_j | j < i\})\}$. By construction and Proposition 1 (iii), $\mathrm{cl}(X \cup Y_0) = \mathrm{cl}(Y)$. Suppose $y_j \in Y_0$ and $y_j \in \mathrm{cl}(X \cup Y_0 - \{y_j\})$. Then $y \in \mathrm{cl}(Y_1)$ where Y_1 is finite, $y \notin Y_2$ for any proper subset Y_2 of Y_1, and $Y_1 \subseteq X \cup Y - \{y_j\}$. Let i be the largest ordinal such that $y_i \in Y_1$. Then $i > j$, since $y_j \in Y_0$. But again we apply Proposition 2 and get $y_i \in \mathrm{cl}((Y_1 - \{y_i\}) \cup \{y_j\})$ which contradicts $y_i \in Y_0$.

Definition 4. Let $Y = \mathrm{cl}(X)$, $X \subseteq M(A)$. The *dimension* of Y, written $\dim(Y)$, is the number of elements in any independent set $Y_0 \subseteq M(A)$ such that $\mathrm{cl}(Y_0) = Y$. Such a Y_0 is called a *basis* for Y.

Corollary 1. (i) $\dim(Y)$ is well defined.

(ii) If $X_1 \subseteq M(A)$, $X_2 \subseteq M(A)$ and $\mathrm{cl}(X_1) \subset \mathrm{cl}(X_2)$ with $\mathrm{cl}(X_1) \neq \mathrm{cl}(X_2)$, then $\dim(\mathrm{cl}(X_2)) \geqslant \dim(\mathrm{cl}(X_1)) + 1$.

This corollary is proved by the same arguments used in proving the corresponding results in linear algebra. We now proceed to some results concerning elementary monomorphisms. We first note that any elementary monomorphism carries independent sets into independent sets.

Proposition 3. If $f : X \to |B|$, $X \subseteq |A|$, is an elementary monomorphism, then f can be extended to an elementary monomorphism $f : \mathrm{cl}(X) \to |B|$ whose range is $\mathrm{cl}(f[X])$.

Proof: $(A, x_i) \equiv (B, f(x_i))$ since f is an elementary monomorphism. Note that $\mathrm{cl}(X) = \mathrm{cl}(\emptyset)$ in (A, x_i). Let ψ_j, $j < \alpha$, be an enumeration of one representative of each atom $[\psi]$ such that $\psi^{(A,x_i)}$ is finite. Then $\mathrm{cl}(X) = \mathrm{cl}(\emptyset) = \bigcup_{j<\alpha} \psi_j^{(A,x_i)}$. Let y_i, $i < \beta$ be an enumeration of $\mathrm{cl}(\emptyset) - X$ such that $n < j$, $y_n \in \psi_k^{(A,x_i)}$ and $y_j \in \psi_m^{(A,x_i)}$, together imply $k \leqslant m$. Let $f(y_0)$ be any element of $\psi_0^{(B,f(x_i))}$. Now $(A, x_i, y_0) \equiv (A, f(x_i), f(y_0))$. Replacing X by $X \cup \{y_0\}$, etc., we get $(A, x_i, y_i) \equiv (A, f(x_i), f(y_i))$ which says that the f is an elementary monomorphism with domain $\mathrm{cl}(X)$. Obviously the range of f is $\mathrm{cl}(f[X])$.

Theorem 2. If ϕ^A is strongly minimal in A, ϕ^B is strongly minimal, X and Y are independent, $X \subseteq \phi^A$, $Y \subseteq \phi^B$ and $f : X \to Y$ is one-one, then f is an elementary monomorphism.

Proof: If ψ^C is a minimal set then every $c \in \psi^C - \mathrm{cl}(\emptyset)$ realizes the same ultrafilter in $B^1(T)$; if ψ^C is strongly minimal, then every $c \in \psi^C - \mathrm{cl}(Z)$ realizes the same ultrafilter in $B^1(\mathrm{th}(C, z_i))$. Let x_i be an enumeration of X. Since ϕ^A and ϕ^B are strongly minimal and X and Y are independent, $(A, x_0) \equiv (B, f(x_0))$ since $x_0 \in \phi^A - \mathrm{cl}(\emptyset)$ and $y_0 \in \phi^B - \mathrm{cl}(\emptyset)$. Also if $(A, x_i)_{i<\alpha} \equiv (B, f(x_i))_{i<\alpha}$ for $\alpha < \mathrm{card}(X)$, then $(A, x_i)_{i<\alpha+1} \equiv (B, f(x_i))_{i<\alpha+1}$; if $(A, x_i)_{i<\alpha} \equiv (B, f(x_i))_{i<\alpha}$ for every $\alpha < \beta$, β a limit ordinal, then $(A, x_i)_{i<\beta} \equiv (B, f(x_i))_{i<\beta}$. Therefore, by induction, $(A, x_i)_{i<\mathrm{card}(X)} \equiv (A, f(x_i))_{i<\mathrm{card}(X)}$.

Definition 5. $X \subseteq |A|$ is *indiscernible* if every one-one $f \in X^X$ is an elementary monomorphism.

Corollary 2. If ϕ^A is strongly minimal, $X \subseteq \phi^A$, and X is independent, then X is indiscernible.

Definition 6. A is *properly imbeddable* in B, written $A < B$ if there is an elementary monomorphism mapping $|A|$ properly into $|B|$.

Note that $<$ is transitive.

Theorem 3. Let T be such that the domain of every model is strongly minimal. Then

(i) T is ω_1-categorical.

(ii) If T is not ω_0-categorical it has ω denumerable models A_i, $i < \omega + 1$ such that every denumerable model is isomorphic to exactly one of them and $i < j$ implies $A_i < A_j$.

(iii) $A < A$ iff $\dim(|A|)$ is infinite.

Proof: (i) Let card(A) = card$(B) = \omega_1$. Choose bases X and Y for $|A|$ and $|B|$ respectively. As noted earlier, card(X) = card$(Y) = \omega_1$. Choose any one-one correspondence f between them and apply Theorem 2 and Proposition 3.

(ii) As above, if $\dim(|A|) = \dim(|B|)$, A and B are isomorphic. If A is denumerable, $\dim(|A|) \leqslant \omega$. If $X \subseteq |C|$ and cl$(X) = X$, then X is a model of T iff X is infinite. For, by the assumptions made about T at the beginning of this chapter, X must be infinite to be a model of T. If X is infinite and cl$(X) = X$, then $\phi^C(v_0, x_1, \dots, x_n)$ is either finite or cofinite in $|C|$. If it is finite, it is in cl$(X) = X$; if it is cofinite in $|C|$ it intersects X. Thus $X \prec C$ (see [6]), and is therefore a model of T. Let card$(C) = \omega_1$. Choose $X = \{x_i\}$, $i < \omega$, $X \subseteq |C|$, with X independent. Let k be the least ordinal such that cl$(\{x_i | i < k\})$ is infinite. If $k = \omega_0$, T is ω_0-categorical. If $k < \omega$ let A_j = cl$(\{x_i | i < k + j\})$ for $j < \omega + 1$. By Theorem 2 and Proposition 3 if $i < j$, $A_i < A_j$.

(iii) If $\dim(|A|)$ is infinite, choose a basis X for $|A|$ and a one-one but not onto $f \in X^X$, and then apply Theorem 2 and Proposition 3. If $\dim(|A|)$ is finite then not $A < A$, by Corollary 1 (ii).

Chapter II

In this chapter we assume that T is a complete theory in a denumerable language which is ω_1-categorical. A, B and C are models of T.

A is a *prime model* for T if for every model B there is an elementary monomorphism of A into B. A is a *saturated model* if for every subset $X \subseteq |A|$ with $\text{card}(X) < \text{card}(|A|)$ every ultrafilter of $B^1(\text{th}((A, x_i)))$ is realized in (A, x_i). Vaught showed in [7] that every T satisfying the assumptions which we made above has $P_n(T)$ denumerable for each n and therefore T has a denumerable saturated model and a prime model, both of which are unique up to isomorphisms; the prime model A is *atomic* and realizes no non-principle ultrafilter in any $B^n(T)$ and it has no proper elementary substructures, i.e., $A' \prec A$ implies $A' = A$.

In [21 Morley defines B to be a *prime extension* of A if $A \prec B$, $A \neq B$ and for any C such that $A \prec C$, there is an elementary monomorphism of B into C which is the identity on A. Using Vaught's Two Cardinal Theorem, Morley observed that if $A \prec B$, $A \neq B$, then $\phi^B - \phi^A \neq \emptyset$ for models A and B of an ω_1-categorical T. He then proved that T is ω_1-categorical iff every denumerable model has a prime extension, and that if T is ω_1-categorical every such prime extension is minimal and any two prime extensions of the same model are isomorphic. Using these results, he has shown (oral communication) that an ω_1-categorical theory has at most ω denumerable models.

In [3] he showed that if T is categorical in one uncountable power it is in all, and every uncountable model is saturated. The proof of Theorem 5.4 in [3] also proves the following theorem, as Professor Morley pointed out to the author: If T is ω_1-categorical and some filter in $B^1(T)$ is not realized in A, then there is no infinite indiscernible subset of $|A|$.

Theorem 4. Let T be ω_1- but not ω_0-categorical, with $B^1(T)$ infinite and $\phi \in L(T)$ defining a strongly minimal set. Then every denumerable model of T is isomorphic to exactly one of an $\omega + 1$ sequence $A_0 < A_1 < A_2 < \ldots < A_\omega$ of models of T.

Proof: Let A_0 be a prime model of T. We assert that $\dim(\text{cl}(\phi^{A_0})) = k < \omega$, for some k. For if not, any basis would be an infinite indiscernible subset of $|A_0|$, and since $B^1(T)$ is infinite it has a non-principle ultrafilter, which is not realized in A_0, because A_0 is prime; this would contradict the last theorem of Morley mentioned above.

Let A_i be a prime extension of A_{i-1} for $0 < i < \omega$ and let $A_\omega = \bigcup_{i<\omega} A_i$. For $i < j \leqslant \omega$, $\phi^{A_j} - \phi^{A_i} \neq \emptyset$, and so by Corollary 1 (ii) $\dim(\text{cl}(\phi^{A_i})) \geqslant k + 1$, for $i \leqslant \omega$. Thus $\dim(\text{cl}(\phi^{A_\omega})) = \omega$, and we contend that equality holds for

$i < \omega$ also. Let $X = \{x_i\}_{i<\omega}$ be an independent subset of ϕ^{A_ω} such that $x_{i+k} \in \phi^{A_{i+1}}$ for all $i < \omega$, and $x_0, \ldots, x_{k-1} \in \phi^{A_0}$. The structure $A'_0 = (A_0, x_i)_{i<k}$ contains a prime model of its theory T'_0, and since A_0 is minimal as a prime model of T, A'_0 is a prime model for T'_0. Let T'_j be the theory of the structure $A'_j = (A_j, x_i)_{i<k+j}$ for $j < \omega$. A'_j is a prime model for T'_j for $j < \omega$, for otherwise, let i be the smallest number such that A'_{i+1} is not a prime model of T'_{i+1}. Let $B'_{i+1} \prec A'_{i+1}$, with $B'_{i+1} = (B_{i+1}, x_0, \ldots, x_{k+1})$, be a prime model for T'_{i+1}. Then $(B_{i+1}, x_0, \ldots, x_{k+i-1})$ is a model of T'_i and contains a prime model B'_i of T'_i, which is therefore isomorphic to A'_i. $x_{k+1} \in |B_{i+1}| - |B_i|$. Thus B_{i+1} is a proper elementary extension of $B_i \cong A_i$. Therefore A_{i+1} can be imbedded in B_{i+1} by an elementary monomorphism f which takes A_i onto B_i; we thus have $f(A_i) \prec f(A_{i+1}) \prec B_{i+1} \prec A_{i+1}$. But then $\phi^{f(A_{i+1})} \subseteq \phi^{B_{i+1}} \subseteq \phi^{A_{i+1}}$ and $\dim(\text{cl}(\phi^{f(A_{i+1})})) = \dim(\text{cl}(\phi^{A_{i+1}}))$, since they are isomorphic. Therefore $\phi^{f(A_{i+1})} = \phi^{B_{i+1}} = \phi^{A_{i+1}}$, and $B_{i+1} = A_{i+1}$, which proves that A'_{i+1} is a prime model of T'_{i+1}. As a prime model of T'_j, A'_j is atomic and ϕ^{A_j} is a union of atoms. $\text{cl}(\phi^{A_0}) = \text{cl}(\{x_0, \ldots, x_{k-1}\})$ because $\dim(\text{cl}(\phi^{A_0})) = k$, so every atom in ϕ^{A_0}, and therefore in $\phi^{A'_j}$ is finite. Thus $\text{cl}(\phi^{A'_j}) = \text{cl}(\emptyset)$ in A'_j. Therefore $\text{cl}(\phi^{A_j}) = \text{cl}(\{x_0, \ldots, x_{k+j-1}\})$, and $\dim(\text{cl}(\phi^{A_j})) = k + j$. Thus all the A_i, $i \leqslant \omega$ are distinct.

If B is a denumerable model of T with $\dim(\text{cl}(\phi^B)) = k + j$ for some $0 \leqslant j < \omega$, then $B' = (B, b_0, \ldots, b_{k+j-1})$ is a model of T'_j and since we have $\dim(\text{cl}(\phi^B)) = k + j$, B' is in fact a prime model of T'_j. Therefore $B' \cong A'_j$ and therefore $B \cong A_j$.

Finally, if B is a denumerable model of T with $\dim(\text{cl}(\phi^B)) = \omega$, let $\{y_i\}$, $i < \omega$, be a basis for ϕ^B with $y_i \in \phi^B$ for all $i < \omega$. $B'_j = (B, y_i)_{i<j+k}$ is a model of T'_j and contains a prime model $C'_j = (C_j, y_i)_{i<j+k}$; thus $C'_j \cong A'_j$ and $C_j \cong A_j$. Let $C_\omega = \bigcup_{i<\omega} C_i$. Then $y_i \in |C_\omega|$ for all $i < \omega$, and since the y_i are a basis, $\phi^{C_\omega} = \phi^B$. Since $C_\omega \prec B$, we have $C_\omega = B$ by Morley's use of the Vaught Two Cardinal Theorem. Thus any denumerable model B with $\dim(\phi^B) = \omega$ is isomorphic to A_ω. Since the denumerable saturated model of T must be such, A_ω is saturated. Morley had observed that A_ω is saturated (oral communication). This completes the proof of Theorem 4.

Finally we wish to prove a result about denumerable saturated models of arbitrary ω_1- but not ω-categorical theories which is complementary to Vaught's result that prime models of such theories are minimal. We first prove that by adding a finite number of constants to a general ω_1- but not ω-categorical theory we can get a theory which satisfies the hypothesis of Theorem 4. It follows immediately from the Ryll-Nardzewski Theorem (see [7] p. 303) that if T is not ω-categorical then by extending T to T' by adding some finite set of constants we can make $B^1(T')$ infinite.

It is easy to see that if every model A of T with card$(|A|)) = \omega_1$ is saturated, so is every model B of th$((A, x_i)_{i<\lambda})$ for any $\{x_i\} \subseteq |A|$ with $\lambda < \omega_1$. Using this fact we can see that by adding a finite number of constants to an ω_1-categorical T, we can define a strongly minimal set. For, if we suppose the contrary, we can partition any infinite set S (of which there must be at least one) into infinite sets S^1_1 and S^1_2 by adding a finite set F_1 of constants; given $S^n_1, S^n_2, \ldots, S^n_{2^n}$, we can partition each into a pair of infinite sets, using some finite set F_n of constants. The theory T' obtained by adding the constants $\bigcup_{i<\omega} F_i$ to T will be ω_1-categorical if T is, by the previous remark, but $P^1(T')$ will be uncountable, which contradicts the result in [7] mentioned at the beginning of this chapter.

Now let T be ω_1- but not ω-categorical and let T' be an extension by a finite set c_i, $i < n$, of constants which satisfies the hypothesis of Theorem 4. Let B be a denumerable saturated model of T. Then B contains an n-tuple $\langle b_0, \ldots, b_{n-1}\rangle$ of the type used in extending T to T', and $(B, b_0, \ldots, b_{n-1}) = B'$ is a model of T'. If $B \prec C$, with card$(|C|) = \omega$, then $(B, b_0, \ldots, b_{n-1}) \prec (C, b_0, \ldots, b_{n-1}) = C'$. The dimension of the closure of the strongly minimal set $\phi^{B'}$ is infinite, therefore so is the dimension of the closure of $\phi^{C'}$. Thus C' is saturated as a model of T', and C, being a reduct of a saturated model is saturated (see [4] p. 50). We have proved

Theorem 5. If T is ω_1- but not ω-categorical, B is a denumerable saturated model of T, and $B \prec C$ with card$(|C|) = \omega$, then C is saturated.

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