Logic

Three tutorials at Tbilisi

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1.1

FIRST TUTORIAL
Entailments

1.2

Two sentences:

(a) Noam Chomsky is very clever.

(b) Noam Chomsky is clever.

If (a) is true then (b) must be true too.
We express this by saying that (a) *entails* (b), in symbols

NC is very clever. $\models$ NC is clever.

1.3

If p and q are sentences, the expression

$$p \models q$$

is called a *sequent*.
If it is true, i.e. if p does entail q,
then we say it is a *valid sequent*,
or for short an *entailment*.

1.4

Provisional definition

Logic is the study of entailments.

But it’s a bad idea to go to a definition so early.

1.5

Typical problem

Imagine a situation with two Noam Chomskys,

one very clever and one very unintelligent.

1.6

Conversation:

- “Noam Chomsky is very clever.”
- “Yes, but on the other hand Noam Chomsky is not clever at all.”

We can understand this conversation.
(David Lewis, ‘Scorekeeping in a language game’.)

1.7

Remedy

A *situation* (for a set of sentences) consists of the information needed to fix who the named people are, what the date is, and anything else relevant to the truth of the sentences.

Then we can revise our notion of entailment:
‘(a) entails (b)’ means

In every situation, if (a) is true then (b) is true.

This remedy is not waterproof,
but it will allow us to continue building up logic.

1.8

We generalise. Suppose $p_1, \ldots, p_n$ and q are sentences. The sequent

$$p_1, \ldots, p_n \models q$$

('$p_1, \ldots, p_n$ entail q') means that in any situation where $p_1, \ldots, p_n$ are all true, q is true too.

$p_1, \ldots, p_n$ are the *premises* of the sequent, q is its *conclusion*.

1.9

In particular

$$\models q$$

expresses that q is true in every situation (i.e. a *necessary truth*).

1.10

Gerhard Gentzen

Gerhard Gentzen made some of the most important discoveries in logic.

In particular he discovered a mathematical theory of entailments,
which we will study for the rest of this lecture.

He deserves to be better known.

1.12

Axiom Rule

For every sentence p,

$$p \models p.$$

For example:

> Bush was right to invade Iraq. $\models$ Bush was right to invade Iraq.

1.13

Three structural rules

1. If $p, q \models r$ then $q, p \models r$. (A typical *exchange* rule)

2. If $p \models r$ then $p, q \models r$. (An example of *monotonicity*, also called *weakening*.)

3. If $p, q, q \models r$ then $p, q \models r$. (A typical *contraction* rule.)

1.14

Jean-Yves Girard (1989):

> In fact, contrary to popular belief, these [structural] rules are the most important of the whole calculus
> ...

Girard in his *linear logic* drops the structural rules in order to study (1) the order in which the premises are used to reach the conclusion, (2) exactly which premises are really needed, and (3) how many times each premise is used.

This makes sense only when we have a notion of 'reaching' the conclusion from the premises.

1.15

The cut rule

Suppose

$$\begin{aligned} p_1, \ldots, p_n &\models q, \\ p_1, \ldots, p_n, q &\models r. \end{aligned}$$

Then

$$p_1, \ldots, p_n \models r.$$

The sentence q is *cut out*.

The cut rule is the only Gentzen rule in which we 'lose' a whole sentence as we generate a new entailment.

1.16

There remain the **Logical rules**.

Typical examples, the rules for 'and':

1 (lefthand rules). If $p \models r$ then

$(p \text{ and } q) \models r$.

If $q \models r$ then

$(p \text{ and } q) \models r$.

2 (rightthand rule). If $p_1, \ldots, p_n \models q$ and $p_1, \ldots, p_n \models r$ then

$p_1, \ldots, p_n \models (q \text{ and } r)$.

1.17

We can justify the rules for ‘and’ by noting that
(p and q) is true exactly when p and q are both true.

Truth table of ‘and’:

p	q	(p and q)
T	T	T
T	F	F
F	T	F
F	F	F

Charles Peirce

1839–1914

introduced truth tables

1.19

But equally we can deduce the truth table of ‘and’ from Gentzen’s rules:

- By the axiom rule, $p \models p$.
  So by the lefthand rule for ‘and’,

  $$(p \text{ and } q) \models p.$$

  So if (p and q) is true then so is p.
  Similarly if (p and q) is true then so is q.

1.20

- Conversely, by the axiom rule, $p \models p$
  so by weakening, $p, q \models p$.
  Similarly $p, q \models q$.
  But then by the righthand rule for ‘and’,

  $$p, q \models (p \text{ and } q).$$

  So if p and q are both true, then so is (p and q).

1.21

Gentzen's rules for 'Every':

3 (lefthand rule):

If

$$(\star\star\star \text{ Noam Chomsky } +\!+\!+) \models r$$

then

$$(\text{For everybody } x,\ \star\star\star\, x\ +\!+\!+) \models r.$$

(We can replace 'Noam Chomsky' by any name of a person, since 'everybody' is a quantifier ranging over people.)

1.22

Comment

We can't replace

$$(\text{For everybody } x,\ \star\star\star\, x\ +\!+\!+)$$

by

$$(\star\star\star \text{ everybody } +\!+\!+).$$

For example

(VALID:) I haven't read Noam Chomsky. $\models$ I haven't read Noam Chomsky.
(INVALID:) I haven't read everybody. $\models$ I haven't read Noam Chomsky.

1.23

4 (righthand rule for 'Everybody'):

If

$$p \models (\star\star\star \text{ Mr or Mrs X } +\!+\!+)$$

then

$$p \models (\text{For everybody } x,\ \star\star\star\, x\ +\!+\!+).$$

Idea: If a statement about Mr or Mrs X must be true regardless of who Mr or Mrs X is,
it must be true about everybody.

1.24

Both of Gentzen's rules for 'Every' follow the language and practice of mathematicians.

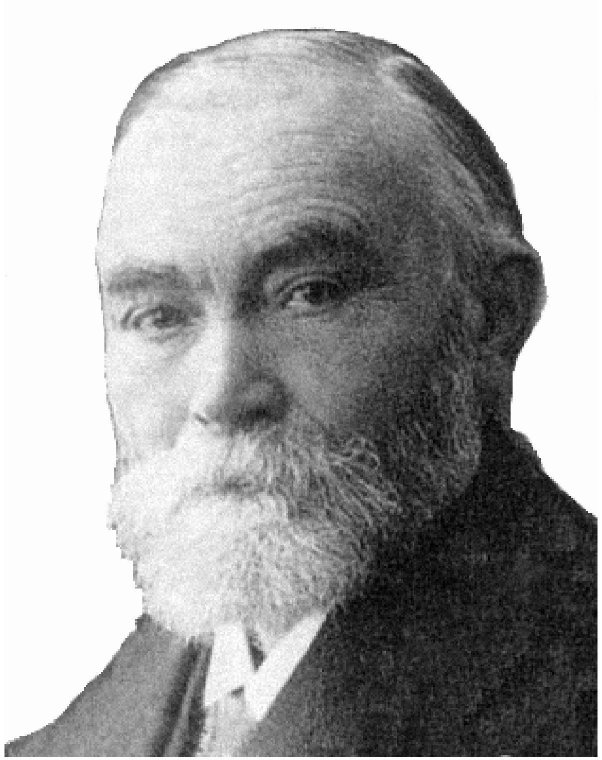
Otherwise he probably wouldn't have discovered them.

Mathematicians had adjusted their language so that entailments could often be recognised from their grammatical structure.
This is a feature of Frege's 'logically perfect languages'.

1.25

Gottlob Frege

1848–1925



1.26

5. Gentzen's righthand rule for 'If … then':

Suppose

$$p, q \models r.$$

Then

$$p \models (\text{If } q \text{ then } r).$$

1.26

Again if you think in terms of 'reaching' the conclusion from the premises,
this rule says that to show you can reach the conclusion

$$(\text{If } q \text{ then } r)$$

it's enough to start from q (i.e. *assume* q)
and then reach r.

1.27

But this is an *interpretation* of Gentzen's rules,
and it lands us in problems about what it is to assume
'for the sake of argument' something we know is false.

One view is that Gentzen's calculus shows how
assumptions 'for the sake of argument' can be explained
without having to consider counterfactual implications.

1.28

To introduce Gentzen's other rules, we have to extend the notions of sequents and entailments, by allowing any number of sentences on the righthand side.

We say the sequent

$$p_1, \ldots, p_n \models q_1, \ldots, q_m$$

is *valid*, and an *entailment*, if there is no situation in which $p_1, \ldots, p_n$ are all true and $q_1, \ldots, q_m$ are all false;
in other words, if in every situation where $p_1, \ldots, p_n$ are all true, at least one of $q_1, \ldots, q_m$ is true.

1.29

6. Gentzen's lefthand rule for 'If ... then':

Suppose

$$p, r \models s$$

and

$$p \models q, s.$$

Then

$$p, (\text{If } q \text{ then } r) \models s.$$

1.30

The link between Gentzen's rules and truth tables is not so clear for 'If ... then' as it is with 'and'.
If 'If ... then' has a truth table, then the Gentzen rules force it to be

p	q	If p then q
T	T	T
T	F	F
F	T	T
F	F	T

1.31

Gentzen's rules for 'not-p' (i.e. 'It is not true that p'):

7 (lefthand). If

$$p \models q, r$$

then

$$p, \text{ not-}q \models r.$$

8 (righthand). If

$$p, q \models r$$

then

$$p \models \text{ not-}q, r.$$

1.32

Notation: We write

$(p \wedge q)$	for	(p and q) ;
$(p \vee q)$	for	(p or q or both) ;
$(p \to q)$	for	(If p then q) ;
$\neg p$	for	not-p ;
$\forall x$	for	(For everything x),
$\exists x$	for	(There exists something x).

First-order logic is logic using just these symbols.

1.33

We can use the Gentzen rules as a mathematical calculus. When we do this, we normally write $\vdash$ rather than $\models$.

Example:
(1) $p, q \vdash q, r$ by axiom rule.
(2) $p, q, r \vdash r$ by axiom rule.
(3) $p, q, (q \to r) \vdash r$ by (1), (2), left rule for $\to$.
(4) $p, (q \to r) \vdash p, r$ by axiom rule.
(5) $p, (p \to q), (q \to r) \vdash r$ by (3), (4), left rule for $\to$.
(6) $(p \to q), (q \to r) \vdash (p \to r)$ by right rule for $\to$.

1.34

We proved:

$$(p \to q), (q \to r) \vdash (p \to r).$$

Gentzen noticed that although the entailment ‘cuts out’ the sentence q, the proof *never uses the cut rule*.

His ‘Cut elimination theorem’ showed that in his calculus, the cut rule is never needed.

Hence we can construct a proof of an entailment by working backwards from the entailment.
(This is the idea behind the tableau calculus.)

2.1

SECOND TUTORIAL
Form and matter

2.2

The definition of logic

Are we ready to come back to this question?

Two naive tests of whether something belongs to logic:

- Do the people who study it call themselves logicians?
- Does it involve the same skills as other things that belong to logic?

2.3

By these tests, there are two clusters of subjects,
one around *entailment* and the other around *definability*,
with important overlaps between them.

Related to entailment:

Proof theory.
Logic programming.
Constructive mathematics and foundations of mathematics.
Modal and temporal logics.
Dynamic logics and logics of processes.
etc.

2.4

Related to definability:

Set theory.
Recursion theory.
Lambda calculus.
Model theory.
Formal semantics of artificial or natural languages.
Formal specification theory.
etc.

Category theory and relation algebras are related to logic
but not in a simple way.
People disagree about whether fuzzy logic is a part of logic.

2.5

This kind of definition of logic draws a picture of an area of research,
but it doesn't answer any fundamental question.

Are there more fundamental questions that could lead us to a more precise definition of logic?

(Myself I believe not, but this is a question that quite a lot of people have strong views on.)

2.6

Entailments are usually instances of general laws

Recall the sequent we proved last time:

(If p then q), (If q then r) $\models$ (If p then r).

The expression

(If p then q)

is not a sentence, because it contains variables p, q.
So we call it a *formula*.

What we showed is that *if we replace p and q by any sentences, then the resulting sequent is a valid entailment*.

2.7

It's useful to generalise the definition of entailment to cover formulas as well as sentences:

If $p_1, \ldots, p_n$ and q are *formulas*, then

$$p_1, \ldots, p_n \models q$$

means:

> Whenever we replace the variables by appropriate expressions so as to get sentences, then the sentences got from $p_1, \ldots, p_n$ entail the sentence got from q.

2.8

Back to our two sentences:

(a) Noam Chomsky is very clever.

(b) Noam Chomsky is clever.

Here (a) entails (b).

Compare:

(a') Georgia is very mountainous.

(b') Georgia is mountainous.

Again (a') entails (b').

2.9

We can generalise to a formula with variables X and Y:

(c) X is very Y.

(d) X is Y.

In our new sense of entailment, (c) entails (d).

2.10

Some people have suggested that when we replace sentences by formulas with variables,
we are 'abstracting away the content'.

Immanuel Kant:

> Logic contains no matter at all, only form of thought.

(From *Dohna-Wundlacken Logic*)

2.11

Immanuel Kant
1724–1804

2.13

According to the textbook of Hilbert and Ackermann, *Grundzuege der Theoretischen Logik* (1928), entailments like this can be handled by logic if we apply 'conceptual analysis'.

Their example was

> There is a son. $\models$ There is a father.

In their conceptual analysis, 'y is a son of x' means 'y is male and x is is either the male parent of y or the female parent of y', and 'x is a father of y' means 'x is the male parent of y'.

2.15

The mathematicians' view:
We can leave conceptual analysis to the philosophers and the linguists.

This is too fast. Recall the sequent we proved earlier:

> (If p then q), (If q then r) $\models$ (If p then r).

2.16

Counterexample (Walter Burley, c. 1300)

p	I call you a donkey.
q	I call you an animal.
r	I state the truth.

2.17

Analysis of Burley's counterexample

The meaning of Burley's r changes according to its context. It expresses 'The statement of mine just mentioned is true'.

We have to put another condition on the sentences used to replace variables in entailments: *Their contribution to the truth or falsity of the sentence as a whole depends only on their truth value in the (non-verbal) situation in question*.

So to apply the sequent calculus to sentences of English, we *must* be prepared to do some conceptual analysis.

2.18

Kant also believed that entailments are related to 'thought'.

We often 'think' entailments

$$p \models q$$

in the sense that we believe p,
and as a result we come to believe q too.

2.19

Some people believe that we have a 'deducing' module,
which we use in all our thinking
and possibly in our use of language too.

Kant again:

> Logical rules are not ones according to which we think, but according to which we ought to think.

2.20

Problem: Work of Peter Wason and others shows that in fact our intuitive reasoning often disagrees with logical entailments.

2.22

The Wason selection task (1966)

You will be shown four cards.

Each of these cards has a number written on one side and a letter on the other.

You will also be shown a statement S about the cards.

Then you must answer the following question:

> Which card or cards must I turn over in order to check whether the statement S is true?

2.23

S: If a card has a vowel on one side, it has an even number on the other side.

E	K	4	7

2.24

The true answer is cards E and 7.

In my logic class (which was already expert with truth tables):

E and 7	0%
E	50%
E and 4	20 %
K and 7	15%
other	15%.

This experiment always gives similar results.

2.25

How did we evolve to be so illogical?

Probable answer: The most important things for the species are

- to avoid being killed by tigers or buses,
- to catch, dig or buy meat or vegetables,
- to seduce members of the opposite sex.

2.26

In these circumstances it is not obvious that we ought always to reason logically.
Logical reasoning is expensive in time and effort.
It uses up large amounts of working memory and requires one to form abstract concepts.

Oaksford and Chater (1994) analysed the Wason selection task in terms of 'optimal data selection', and got a prediction not far from the observed facts.

2.27

Summary

Logicians can often explain entailments as examples of general laws.

One can develop these general laws as a mathematical theory.
In a moment we shall see that one can not only prove laws, but also refute incorrect laws of entailment.

But we hit serious practical and philosophical questions when we apply the general laws to either (a) entailments in ordinary English or (b) intuitive human reasoning.

2.28

Eliminating false entailments

We can show that a rule is false by giving a counterexample.

For example Aristotle seems to have believed the following entailment:

If p then q. If not-p then q. $\models$ Not-q.

Counterexample:

p	Rome is beautiful.
q	Rome is a town.

2.29

Euclid's parallel postulate

Euclid wrote some axioms for geometry, including one known as the Parallel Postulate.

(A modern form of the Parallel Postulate: For every infinite straight line L in the plane and every point P not on L, there is exactly one infinite straight line in the plane that passes through P and has no points in common with L.)

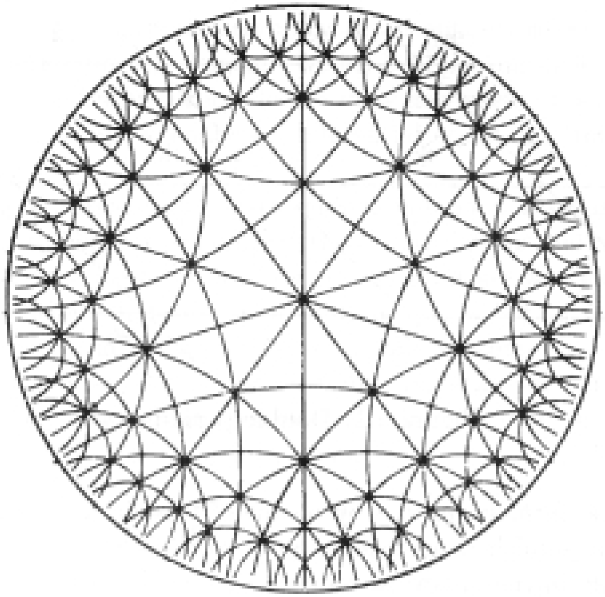
An old question: Do Euclid's other axioms together entail the Parallel Postulate?

2.30

19th century mathematicians gave the answer No, by treating the terms 'point', 'line', 'passes through' etc. as variables, and then replacing these variables by other expressions so as to make the Parallel Postulate false but the other axioms true.

2.31

Schwarz's tesselation
(1872)



2.32

Hence the question: If an entailment is written in the language of Gentzen's sequences, what is the relation between

(a) The sequent is provable in the sequent calculus.

(b) There is a counterexample to the sequent?

The *completeness theorem* (essentially Goedel, 1930) says that for first-order logic, exactly one of (a) and (b) holds.

3.1

THIRD TUTORIAL
Truth and proof

3.2

We saw that we can use the Gentzen sequent calculus to prove valid sequents in first-order logic.

By the completeness theorem, we can prove all and only the valid sequents this way.

There are several other proof calculi with various strong and weak points.
For example Gentzen's *natural deduction* calculus uses proofs that (roughly) start from the premises and finish with the conclusion.

3.3

Propositional logic is first-order logic using only $\wedge$, $\vee$, $\rightarrow$ and $\neg$, not $\forall$ or $\exists$.

We can use truth tables as a kind of proof calculus for propositional entailments.

3.4

Example

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$(p \rightarrow r)$
T	T	T	**T**	**T**	**T**
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	**T**	**T**	**T**
F	T	F	T	F	T
F	F	T	**T**	**T**	**T**
F	F	F	**T**	**T**	**T**

3.5

Truth tables are more than a proof calculus for propositional entailments.

They form a *decision procedure*.
This means we can use truth tables to check mechanically whether or not any given propositional sequent is valid.

3.6

Many mathematical problems can be solved by truth tables, by *translating* them into propositional logic.

Example. A *proper colouring* of a map is a colouring of the countries in the map, so that if two countries have a border in common then they are coloured different colours.

In 1976 Appel and Haken used a computer to prove a very old conjecture: *Every map on the surface of a sphere has a proper colouring using at most four colours.*

3.7

Suppose we have a map and we want to know whether it has a proper colouring with three colours (R, G, B).

Just trying to colour it is not a good strategy.
We will almost certainly have to keep changing colours as we go.

3.8

Instead we number the countries,
and for each country i and each colour C we write

$$p_{iC}$$

for the statement:

Country i has colour C.

3.9

Requirements

(1) For every country i,

$$\neg(p_{iR} \wedge p_{iG}),\ \neg(p_{iR} \wedge p_{iB}),\ \neg(p_{iG} \wedge p_{iB}).$$

(2) For every country i,

$$((p_{iR} \vee p_{iG}) \vee p_{iB}).$$

(3) For every pair of countries i and j with a common border,

$$\neg(p_{iR} \wedge p_{jR}),\ \neg(p_{iG} \wedge p_{jG}),\ \neg(p_{iB} \wedge p_{jB}).$$

3.10

Now we draw a truth table showing all possible values of the propositional variables p_{iC}, and we look for a row in which all the formulas (1), (2), (3) have the value True.

This row tells us how to colour the map:
if p_{iC} is true in the row, colour country i colour C.

3.11

Problem.

Suppose our map has 100 countries.
Then there are 300 propositional variables,
and the number of rows in the truth table is

$$2^{300},\ \text{which is about } 10^{90}.$$

The size of the calculation increases exponentially with the number of countries.

3.12

In 2000 the Clay Mathematics Institute offered a prize of a million dollars to anybody who can show whether there are mechanical methods for solving truth table problems of this kind in a reasonable time.

This is the ‘P = NP’ problem.

3.13

Truth table calculations

We calculate the truth table for one of the sentences in (1) of slide 3.9:

p_{1R}	p_{1G}	$\neg$	$(p_{1R}$	$\wedge$	$p_{1G})$
T	T		T		T
T	F		T		F
F	T		F		T
F	F		F		F

We write under each letter its truth value in each row.

3.14

Next we join p_{1R} and p_{1G} with $\wedge$,
and we write the truth values for $(p_{1R} \wedge p_{1G})$ under $\wedge$:

p_{1R}	p_{1G}	$\neg$	$(p_{1R}$	$\wedge$	$p_{1G})$
T	T		T	T	T
T	F		T	F	F
F	T		F	F	T
F	F		F	F	F

3.15

Finally we add $\neg$ at the left,
and we write the truth value of the whole formula under $\neg$:

p_{1R}	p_{1G}	$\neg$	$(p_{1R}$	$\wedge$	$p_{1G})$
T	T	**F**	T	T	T
T	F	**T**	T	F	F
F	T	**T**	F	F	T
F	F	**T**	F	F	F
		$\Uparrow$			

3.16

In this calculation we calculate the truth values,
starting from the sentence letters
and working up to more and more complex formulas.
The value of a compound formula is determined by the values of its immediate constituents,
including the symbol ($\wedge$, $\neg$) used to combine them.

So the assignment of truth values is *compositional*.

3.17

In 1933 Alfred Tarski showed how to extend these calculations to logic with quantifiers.

On the left, instead of truth value assignments we have structures
(in general infinitely many)
and assignments of elements of these structures to the free variables.

The ‘value’ of a formula is the class of those structures and assignments that make it true.
These values are still assigned compositionally.

3.18

Suppose that instead of the formulas of first-order logic, we consider meaningful sentences of a natural language. Then in place of structures we can take ‘possible worlds’, and we can follow the suggestion of Carnap and Quine that Tarski’s way of assigning truth values is a good substitute for assigning *meanings*.

The result is Montague’s semantics for fragments of English. The semantics is compositional, following Tarski’s example.

3.19

Testing validity

For full first order logic we don’t have a way of testing whether any given entailment is valid.

In fact Alonzo Church showed in 1936 that there can’t be such a method, either fast or slow.

3.21

G. W. Leibniz

1646–1716

3.22

Leibniz hoped to build a calculus (a ‘Universal Characteristic’) that would allow us to solve problems of reasoning by pure calculation.

Important point: He never claimed his calculus would determine whether a given statement is true.

He did claim that his calculus would determine whether any proposed proof is correct or not.

3.23

Three questions:

(1) Is this a proof?

(2) Does this have a proof?

(3) Is this true?

Apparently Leibniz foresaw that question (1) is decidable, but he made no claim about (2) or (3).

3.24

In 1931 Kurt Goedel showed that the question whether a given arithmetical statement is true is an undecidable question.

In 1970 he wrote to a student (Yossef Balas) saying that the key to proving this was to understand the difference between truth and proof.

We shall sketch a proof of Church’s theorem that rests on the same idea.

3.25

We consider the structure $\mathbb{N}$ consisting of the natural numbers.

$$0, 1, 2, 3, \ldots$$

We write symbols S for ‘plus one’, and 0 for the number zero.

We write down some true statements about $\mathbb{N}$, using these symbols:

3.26

If x and y are numbers with $S(x) = S(y)$ then $x = y$.
There is no number x such that $0 = S(x)$.
If $x \neq 0$ then there is y such that $x = S(y)$.
For every number x, $x + 0 = x$ and $x.0 = 0$.
For all numbers x and y, $x + S(y) = S(x + y)$ and $x.S(y) = x.y + x$.

The set of these five statements is called Q.

3.27

By a *diophantine sentence* we mean a sentence of the form

There are numbers $x_1, \ldots, x_n$ such that $s = t$.

where s and t are arithmetical expressions using $x_1, \ldots, x_n$, 0, S, $+$ and $.$.

Example. The sentence

There are numbers x, y, z such that $x^2 + y^2 = 4z + 3$.

is a diophantine sentence,
writing $SSS(u)$ for $u + 3$ and $SSSS(0)$ for 4.

This sentence happens to be false.

3.28

In 1888 Richard Dedekind showed (in effect) that for every diophantine sentence p, p is true if and only if Q entails p.

So truth and provability agree up as far as diophantine sentences.

Goedel's intuition was that this agreement must break down for more complex sentences, and it's just a matter of finding how far we have to search for the breakdown.

3.29

Goedel and Yuri Matiyasevich, developing ideas of Leibniz, showed that if we use numbers to label the symbols in a proof, then we can express and prove properties of proofs by using diophantine sentences.

In particular, *if we have a mechanical test for whether any given diophantine sentence is true*, then for every number n the sentence

The diophantine sentence with number n is false.

can be written as a diophantine sentence $\theta(n)$.

3.31

Goedel showed also that if this formula $\theta(x)$ exists, then we can find a number n such that the sentence with number n is in fact

$$\theta(n).$$

So $\theta(n)$ expresses that $\theta(n)$ is false.
But this is impossible, since $\theta(n)$ is true if and only if $\theta(n)$ is false.
(Compare the Liar paradox: ‘This sentence is not true’.)

So there is test for whether any given diophantine sentence is true.

3.32

It follows that there is no mechanical test for whether or not a sequent of first-order logic is a valid entailment.

For if there was such a test,
we could use it to test whether or not a given diophantine sentence θ is true,
by checking whether or not $Q \vdash \theta$.