# NON-STRUCTURE 2

Given a class $(K)$ of structures, an *invariant function* on $\mathbf{K}$ is a function $\Gamma$ with domain $\mathbf{K}$ such that

$$M \cong N \;\Rightarrow\; \Gamma(M) = \Gamma(N).$$

We call $\Gamma$ *faithful* if '$\Leftrightarrow$' holds instead of '$\Rightarrow$'.

*Motivating example*: $\mathbf{K}$ is the class of algebraically closed fields $M$, and $\Gamma(M) =$

$\langle$characteristic$(M)$, transcendence degree$(M)\rangle$.

We write $I(\lambda, \mathbf{K})$ for the number of isomorphism classes of structures in $\mathbf{K}$ of cardinality $\lambda$, i.e. the size of the range of a faithful invariant function restricted to structures in $\mathbf{K}$ of cardinality $\lambda$.

We call a class $\mathbf{K}$ *bad* if $I(\lambda, \mathbf{K}) = 2^\lambda$ (the maximum possible value) for all large enough $\lambda$.

Recall that for any unsuperstable complete first-order theory $T$ the class of models of $T$ is bad.

If $\mathbf{K}$ is bad, this is reckoned to be evidence that $\mathbf{K}$ has no good structure theory.
We shall discuss this.

*Default assumption*: A class $\mathbf{K}$ is the class of all models of a complete first-order theory in a countable language.

If $\mathbf{J}, \mathbf{K}$ are classes of structures and there is a map from $\mathbf{J}$ to $\mathbf{K}$ which preserves non-isomorphism and cardinality on infinite structures, then $\mathbf{J}$ bad implies $\mathbf{K}$ bad.

*Example*: Let $L$ be a first-order language with finite signature.
Then the class $\mathbf{J}$ of $L$-structures is faithfully interpretable in the class $\mathbf{Graph}$ of simple graphs (i.e. graphs with no double edges or loops).
This gives a mapping from $\mathbf{J}$ to $\mathbf{Graph}$ which preserves non-isomorphism and cardinality on infinite structures.
(Loewenheim 1915, Lavrov 1963; see Hodges, Model Theory §5.5.)

So by the previous lecture, using a suitable $\mathbf{J}$, the class $\mathbf{Graph}$ is bad.

*Example*: A theory $T$ with DOP ('dimensional order property')

Typical model $M$ is a bipartite graph with parts $P, Q$, both infinite; for each pair $b_1 \neq b_2$ of elements in $P$ there are infinitely many vertices in $Q$ joined to both $b_1$ and $b_2$, and each element in $Q$ is joined to exactly two elements in $P$.

We code up any infinite graph $G$ as a model $M_G$.
In $M_G$ the elements of $P$ are the vertices of $G$.
For any distinct vertices $a, b$ of $G$ we
put in $\omega_1$ elements of $Q$ joined to them both if $a, b$ are joined in $G$, and $\omega$ elements if $a, b$ are not joined in $G$.

The map $G \mapsto M_G$ preserves cardinality and non-isomorphism, and **Graph** is bad. So (the class of models of) $T$ is bad.

Shelah isolated the feature of $T$ which makes it bad. Complete first-order theories with this feature are said to have *DOP*; those without it have *NDOP*.

Given sets $B \subseteq C$ of elements of a model, let $p$ be a (complete) type over $C$. We say $p$ is *orthogonal to* $B$ if $p$ is orthogonal to every type over $C$ which doesn't fork over $B$.

The defining property of DOP (cf. Lascar 1985): There are sets $A, B_1, B_2$ in a model, with $A \subseteq B_1 \cap B_2$ and $B_1, B_2$ independent over $A$, and a type $p$ over $A$, such that $p$ is orthogonal to $B_1$ and to $B_2$ but not to $B_1 \cup B_2$.

A theory $T$ has the *OTOP* (the Omitting Types Order Property) if there is a type $p(\bar{x}, \bar{y}, \bar{z})$ such that for every $\lambda$ and every 2-ary relation $R$ on $\lambda$, there is a model $M$ of $T$ with elements $\bar{a}_i$ $(i < \lambda)$ such that for all $i, j < \lambda$,

$$iRj \;\Leftrightarrow\; p(\bar{a}_i, \bar{a}_j, \bar{x}) \text{ is realised in } M.$$

A theory without the OTOP has the *NOTOP*.

Examples of OTOP without DOP are not simple to describe.

*Example*: a deep theory

$F$ a 1-ary function symbol, $c$ a constant. The theory $T$ says:

$$\forall x\ (F^n(x) = x \leftrightarrow x = c) \quad (n > 0)$$

$$\forall x \exists_{\geq n} y\ F(y) = x \quad (n < \omega).$$

Define the rank of an element $a$ in model $M$:

rank($a$) $\geq 0 \Leftrightarrow |F^{-1}(a)| \geq \omega_1$.

rank($a$) $\geq \gamma + 1 \Leftrightarrow$ there are uncountably many $b$ of rank $\geq \gamma$ in $F^{-1}(a)$.

rank($a$) $\geq \delta$ (limit) $\Leftrightarrow$ rank($\alpha$) $\geq \gamma$ for all $\gamma < \delta$.

For any nonempty subset $Y$ of a cardinal $\lambda$, make a model $M_Y$ by putting immediately above element $c$ elements of just the ranks in $Y$. This gives $2^\lambda$ models of cardinality $\lambda$.

Shelah isolated the feature of this example that makes it bad.

If $T$ is superstable without DOP, then enough-saturated models of $T$ have a tree structure, which can be ranked like the example above.

The *depth* of $T$ is the least upper bound of the ranks of models.
We say $T$ is *deep* if its depth is $\infty$, or equivalently, $\geq \omega_1$.
We say $T$ is *shallow* if its depth is at most countable.

*Shelah's Main Gap* (for countable superstable theories)

| | DOP or OTOP | NDOP and NOTOP |
|---|---|---|
| Deep | **Bad** | **Bad** |
| Shallow | **Bad** | good |

*WARNING*. If $T$ is superstable without DOP or OTOP, and depth($T$) $\geq$ 2, then for every infinite $\alpha$,

$$I(\omega_\alpha, T) \geq \min(2^{\omega_\alpha}, 2^{|\alpha|}).$$

There is a closed unbounded class $C$ of cardinals

$$\lambda = \omega_\alpha = |\alpha|,$$

so for any $\lambda$ in $C$,

$$I(\lambda, T) = 2^\lambda$$

making $T$ bad on a closed unbounded set.

Shelah (1985): 'Thus if one is able to show that the theory has $2^{\aleph_\gamma}$ models of power $\aleph_\gamma$ this establishes non-structure.'

*Question*: Does the argument in the case of deep theories show non-structure, or just many models?

To make this a question in mathematics and not in philosophy, one should:

- look at well-established structure theorems,
- isolate mathematical features which make these structure theorems good,
- try to see what classes of structures have these features.

*Example of structure theorem*: Totally projective abelian $p$-groups for a fixed prime $p$ (Fuchs, Infinite Abelian Groups II Chapter XII)

An abelian $p$-group $A$ is *totally projective* if for all ordinals $\alpha$ and all abelian groups $C$,

$$p^{\alpha}\text{Ext}(A/p^{\alpha}A, C) = 0.$$

The *Ulm-Kaplanskysequence* $\Gamma(A)$ of an abelian $p$-group $A$ of cardinality $\leq \lambda$ (infinite) is a well-ordered sequence of length $< \lambda^{+}$; its terms are the dimensions of certain $\mathbb{F}_p$-vector spaces extracted from $A$.

The structure theorem of Crawley, Hales and Hill says that two totally projective abelian $p$-groups are isomorphic if and only if they have identical Ulm-Kaplansky sequences.

NB: The class of totally projective abelian $p$-groups is bad.

The Ulm-Kaplansky sequence of a totally projective abelian $p$-group $A$ of cardinality $\lambda$ is determined by the $L_{\lambda+,\lambda}$-theory $\mathsf{Th}_{\lambda+,\lambda}(A)$ of $A$.

This suggests a new notion of bad class:
$\mathbf{K}$ is bad' if it contains two structures $A, B$ of cardinality $\lambda$ such that

$$A \not\cong B, \quad \mathsf{Th}_{\lambda+,\lambda}(A) = \mathsf{Th}_{\lambda+,\lambda}(A).$$

A theory is called *classifiable* if it is unsuperstable and has NDOP and NOTOP, *unclassifiable* otherwise.

Shelah (1987 and Classification Theory, Theorem XIII.1.1): The following are equivalent, for any countable theory $T$ and any cardinal $\lambda > 2^{\omega}$:

- $T$ is classifiable.
- Any two $L_{\infty,\lambda}$-equivalent models of $T$ of cardinality $\lambda$ are isomorphic.

Have we drawn the class of bad' structures too narrowly?

The Ulm-Kaplansky invariants of a totally projective abelian $p$-group have other good properties, e.g. they are absolute under extensions of the set-theoretic universe that fix cardinalities (such as ccc forcing).

Satisfying a fixed sentence of $L_{\infty,\lambda}$ is not necessarily preserved under ccc forcing.
For example when $\lambda > \omega$ we can express that a model of second-order number theory contains only constructible sets.

Baldwin, Laskowski and Shelah (1993): If $T$ is unclassifiable then there are two nonisomorphic models of $T$ that can be made isomorphic by ccc forcing.

Certain classifiable theories have this property too!

Laskowski and Shelah (1996): If $T$ is superstable but not $\omega$-stable, and has at most countably many $n$-types over $\emptyset$ for each $n$, then by ccc forcing we can create two models of $T$ that are nonisomorphic but can be made isomorphic by further ccc forcing.

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