Classification over a predicate

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Some ways in

Wilfrid Hodges, *Model Theory*, Cambridge UP 1993, section 12.5 Relative Categoricity.

Anand Pillay and Saharon Shelah, *Classification over a Predicate I*, Notre Dame Journal of Formal Logic 26 (1985) 361–376.

Saharon Shelah, *Classification over a Predicate II*, in Shelah, *Around Classification of Models*, Lecture Notes in Mathematics 1182, Springer 1986.

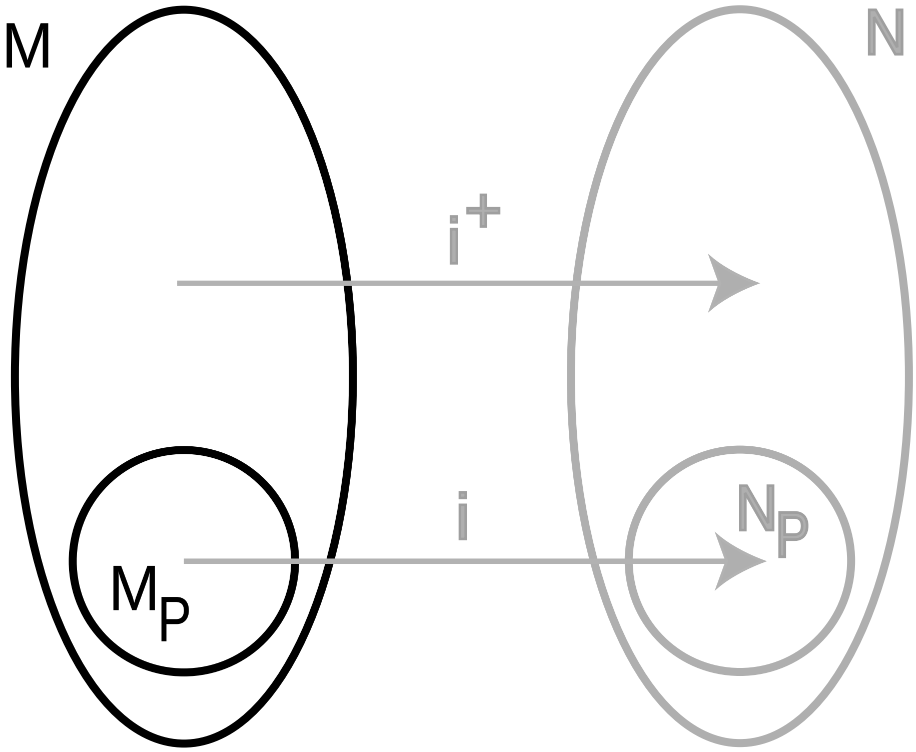
(Note also Shelah 322 unpublished, *Classification over a Predicate*, 1989.)

David Evans, Dugald Macpherson and Alexander Ivanov, *Finite Covers*, in Evans ed., *Model Theory of Groups and Automorphism Groups*, Cambridge UP 1997.

T is a complete theory in a first-order language L^+, one of whose symbols is the 1-ary relation symbol P.

L is a first-order language $\subseteq L^+$ not containing P.

Assumption One: For every model M of T,
the L-reduct of M has a substructure whose elements form the set P^M;
we call this substructure M_P, in words the
P-*part* of M.



Assumption Two (Relative categoricity): If $M \equiv N$ and $M_P = N_P$ then there is an isomorphism i^+ from M to N which is identity on M_P.

Equivalent: If $M \equiv N$ then every isomorphism $i : M_P \to N_P$ extends to an isomorphism $i^+ : M \to N$.

When $T \vdash \forall x \ Px$, relative categoricity reduces to implicit definability (Beth).

When $T \vdash \forall x \ \neg Px$, it reduces to the uninteresting notion of categoricity,
but as usual we can ask about models of a particular cardinality κ and then it reduces to κ-categoricity.

Either way, the natural questions are about definability of M in terms of M_P.

Theorem (Gaifman 1974). Suppose T is rigidly relatively categorical (i.e. the isomorphism i^+ in Assumption Two is unique),
M_P is always infinite
and L^+ is countable.
Then M is uniformly definable over M_P, i.e.

- Every element of M is definable in M with parameters in M_P;
- (Uniform Reduction, = Shelah's Hypothesis 1.1) For every $\phi(\bar{x})$ in L^+ there is $\phi^P(\bar{x})$ in L such that for all models M of T and tuples $\bar{a}$ in M_P,

$$M \models \phi(\bar{a}) \;\Leftrightarrow\; M_P \models \phi^P(\bar{a}).$$

Much work has been about deriving versions of Gaifman's conclusions from weaker hypotheses.

Uniform reduction

We say T is (λ, μ)-*categorical* if

- T has (λ, μ)-models, i.e. models M with $|M_P| = \lambda$ and $|M| = \mu$.
- If M, N are (λ, μ)-models of T with the same P-part, then there is an isomorphism from M to N which is identity on the P-part.

Thus relative categoricity and 'P-part always of cardinality at least $|L^+|$' implies (λ, λ)-categoricity for all $\lambda \geq |L^+|$.

Theorem (Pillay and Shelah). If T is (λ, λ)-categorical for some infinite λ then T has uniform reduction.

But in fact

Theorem. If T is the theory of an abelian group with a subgroup picked out by P, and T is (λ, μ)-categorical for some λ, μ, then T has uniform reduction.

The proof uses Feferman-Vaught properties of modules; how far does the result extend?

Defining elements of M over M_P

If M is not rigid over M_P, Gaifman's definability result obviously fails.

But we can still ask whether M is a reduct of a definable extension of M_P in Gaifman's sense.

Answer: No.

Trivially every automorphism of M restricts to one of M_P,

$$\sigma : \mathrm{Aut}(M) \to \mathrm{Aut}(M_P).$$

By the relative categoricity property,
there is a one-sided set-theoretic inverse

$$\iota : \mathrm{Aut}(M_P) \to \mathrm{Aut}(M), \quad \sigma\iota = 1_{M_P}.$$

If M is a reduct of a definable extension of M_P then
ι can be chosen to be a group homomorphism,
i.e. σ is a split surjection.

Then there are easy counterexamples, e.g.

$$M = \bigoplus_{\omega} \mathbb{Z}/4\mathbb{Z},$$

$$M_P = \bigoplus_{\omega} \mathbb{Z}/2\mathbb{Z}.$$

For proof that σ doesn't split in this case, see the literature on covers, e.g.

Gisela Ahlbrandt and Martin Ziegler, *What's so special about* $(\mathbb{Z}/4\mathbb{Z})^{(\omega)}$*?*, Archive for Mathematical Logic 31 (1991) 115–132.

David Evans, Wilfrid Hodges and Ian Hodkinson, *Automorphisms of bounded abelian groups*, Forum Mathematicum 3 (1991) 523–541.

We can also ask whether in general

(Shelah's Hypothesis 1.2) Every type over M_P in M is definable with parameters in M_P;
i.e. for every type $p(\bar{x})$ over M_P and every formula $\phi(\bar{x}, \bar{y})$ of L there is $\phi^\star(\bar{y})$ with parameters from M_P such that whenever $\bar{b}$ satisfies p,

$$M \models \phi(\bar{b}, \bar{c}) \;\Leftrightarrow\; M \models \phi^\star(\bar{c}).$$

This is guaranteed if T is a stable theory.

Theorem (Shelah). If T is relatively categorical then Hypothesis 1.2 holds.

In fact Shelah proves much more along the same lines, often using set-theoretic assumptions.

Transfer results

Saharon Shelah and Bradd Hart, *Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$*, Israel Journal of Mathematics 70 (1990) 219–235. (Subtitled 'To make Leo happy'.)

If T is the theory of an abelian group with P picking out a subgroup,
then all reasonable transfer theorems hold, e.g.
(λ, μ)-categoricity with $\omega \leq \lambda < \mu$ implies
(λ', μ')-categoricity whenever $\omega \leq \lambda' < \mu'$.

But the proof (Hodges, unpublished) uses a lot of abelian group theory–e.g. Kaplansky-Mackey theorem, cotorsion groups.

So we have little idea what general theorems about transfer can be proved.

For T describing pairs of abelian groups as above, Anatolii Yakovlev has unpublished notes on which pairs give rise to relatively categorical theories. (Not many do.)