Model theory of pairs of abelian groups

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T is henceforth a complete first-order theory.

Baldwin and Lachlan (1971):
If λ is an uncountable cardinal
and all models of T of cardinality λ are isomorphic,
then every model A is determined
by a subset definable (with parameters $\ldots$)
called the *strongly minimal set*.

Gaifman (1974):
What does it tell us if every model A of T is determined
up to isomorphism over its P-part
(i.e. substructure A^P picked out by relation symbol P)?

If language is countable and A is always rigid over A^P,
then A is explicitly definable in A^P (in an obvious sense).

Drop rigidity and countability, and things become
very much harder.

We say the complete theory T is (relatively) (κ, λ)-*categorical*
if it has models A with

$$|A^P| = \kappa, \ \ |A| = \lambda,$$

and if B is another such model, then every isomorphism
$i : A^P \to B^P$ extends to an isomorphism $j : A \to B$.

Relative categoricity is harder than ordinary categoricity.

(1) We can't use an Ehrenfeucht-Mostowski
argument to count types,
unless we know that we can realise new types
without increasing the P-part.

(2) In any case, building up A over A^P,
we have to omit the type 'new element of P-part'.

So we have to find ways of omitting this type,
without having ways to guarantee even that T is stable.

Shelah (various papers, mostly unpublished)
attacks the question using his 'abstract elementary classes' approach:

- Many-dimensional amalgamations over countable submodels,
- strong set-theoretic assumptions to get many models non-isomorphic over P-part when amalgamations fail.

Shelah: 'We expect that the solution will be long, involving many branches.'
Leo Harrington: 'Why is it all so hard?'

Hart, Shelah: 'Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$', Israel J. Maths 70 (1990) 219–235.
(Unofficial subtitle: 'To make Leo happy')

Shelah, Villaveces, 'Categoricity may fail late', arXiv 14 April 2004.

Survey in Rami Grossberg, 'Classification theory for abstract elementary classes', *Logic and Algebra*, ed. Yi Zhang, American Mathematical Society, Providence RI 2002, pp. 165–204.

Although relative categoricity is about pairs of structures
with one a defined substructure of the other,
no connection has appeared yet
with the stability work on pairs:

e.g.
Poizat and Bouscaren on beautiful pairs,
Baldwin and Benedikt on embedded finite models.

My own involvement:

Early on I decided to try to find assumptions under which most of Shelah's complications would disappear.

I haven't succeeded (yet).
For many years I got stuck classifying the (κ, λ)-categorical pairs consisting of an abelian group with P-part a subgroup.

Not in principle hard, but hard to keep track while being dean. My thanks to Ian Hodkinson and Anatoliĭ Yakovlev for helping me not give up.

Partial results had some useful applications. For example

Theorem There is no set-theoretic formula which, provably from ZFC, defines for each field F an algebraic closure of F.

The proof has two parts, a set-theoretic and an algebraic.
The set-theoretic, due to Shelah, uses field extensions with certain automorphism groups.
Calculations with relatively categorical pairs of groups, plus some Galois Theory, found the required fields.
(Oviedo Proceedings, forthcoming.)

Recently I went back to the abelian groups and cleaned up.

In Shelah's classification we are at the very bottom level;
A is 'ω-stable over A^P'.

By Macintyre, an abelian group is ω-stable
if and only if it's infinite and divisible-plus-bounded
(i.e. a sum of a divisible group and a bounded group—such a sum always splits).

There are two main cases:

(a) Both A and A^P are divisible-plus-bounded.
Then A/A^P is also divisible-plus-bounded.

(b) A^P is arbitrary and A/A^P is bounded.

So a key step is to show that if A/A^P is not divisible-plus-bounded,
this prevents (κ, λ)-categoricity.

Main idea:

If A is not divisible-plus-bounded,
then for some group B elementarily equivalent to A,
there is a non-split short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow \mathbb{Q} \longrightarrow 0$$

(i.e. B is not cotorsion).

Make the sequence into a structure and take an ω_1-saturated elementary extension of it. This gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \mathbb{Q}^{(\mu)} & \longrightarrow & 0 \end{array}$$

Here B' is ω_1-saturated, hence pure-injective.
Since $\mathbb{Q}^{(\mu)}$ is torsion-free, B' is pure in C'.
So the bottom sequence splits.

Now

$$B \equiv B' \equiv B' \oplus \mathbb{Q}^{(\mu)} = C' \equiv C.$$

So we can replace B by C and still have a model of T.
(Adding direct summand $\mathbb{Q}^{(\mu)}$ never affects
the theory of an unbounded group.)

The Feferman-Vaught theorem for direct products
(including direct sums $A \oplus B$)
allows us to make this replacement
when B is a direct summand in another group.

When B lies outside the P-part,
we can make this adjustment without affecting the P-part,
and so violate categoricity over P.

When A^P itself is not cotorsion, we put

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^P & \longrightarrow & C & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A^+ & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \end{array}$$

with the left square a pushout.
We define $(A^+)^P = A^P$. Then $A \equiv A^+$ as group pairs.

In fact the idea above doesn't quite work, and we adjust it.

We choose B not cotorsion, and
realising countably many types over any countable subset.
Then we extend B to C with $C/B = \mathbb{Q}^{(\mu)}$,
where C realises uncountably many types
over a countable subset.

There are two kinds of case, according as B has

- pure subgroup $\mathbb{Z}_{(p)}$ (the rationals without q in denominator),
- $\bigoplus_{p_i} \mathbb{Z}(p_i^{k_i})$ with infinitely many distinct primes p_i.

Sketch of second case: Assume B is a countable pure subgroup of

$$\mathbb{Z}(p_0) \times \mathbb{Z}(p_1) \times \mathbb{Z}(p_2) \times \ldots$$

containing $\bigoplus_{p_i} \mathbb{Z}(p_i^{k_i})$.
We realise another type over $\bigoplus_{p_i} \mathbb{Z}(p_i^{k_i})$
by adding another element a of the product.

We iterate this ω_1 times.
Problem is to do it so that the quotient each time is $\mathbb{Q}$.

We have to add a,
and for each $n \geqslant 2$ an element equal to a/n
everywhere except at finite number of coordinates,
so that every integer multiple of
each of these countably many elements
disagrees with each element of B at some coordinate .

There are infinitely many coordinates and
countably many tasks.
So we can schedule the tasks
and eventually complete each one.

When A/A^P is bounded, this construction is impossible.
Hence in this case there is no constraint on A^P at all.

Either way, A/A^P is always divisible-plus-bounded.
This allows us to decompose A as $C \oplus D$ where
$A^P \subseteq C$ and C is *tight* over A^P,
i.e. there is no subgroup G of C disjoint from A^P
with $G/(G + A^P)$ pure in C/A^P.

This means (among other things) that the Ulm-Kaplansky invariants of C over A^P are zero,
so the finite Ulm-Kaplansky invariants over 0 in D
are determined by T.

In all cases, (κ, λ)-categoricity guarantees
the Reduction Property:
given any $\phi(\bar{x})$ in $L(P)$ there is $\phi^\star(\bar{x})$ in L such that
for every model A of T and every $\bar{a}$ in A^P,

$$A \models \phi(\bar{a}) \;\Leftrightarrow\; A^P \models \phi^\star(\bar{a}).$$

When models of T are finite, this just says that every automorphism of A^P extends to an automorphism of A.

The Reduction Property tells us that
if A, B are two models of T
with $i : A^P \to B^P$ an isomorphism,
then i preserves finite p-heights in A and B, for all primes p.

This allows us to use the Kaplansky-Mackey extension lemma to extend i to the summands of A and B that are tight over A^P, B^P.

Other arguments (depending on κ and λ) extend the isomorphism to the second summands in A, B.
Thus a group-theoretic description plus the Reduction Property characterises (κ, λ)-categoricity.

The group-theoretic descriptions don't distinguish between uncountable cardinals.

Hence a Morley theorem:

Theorem If T is (κ, λ)-categorical for some infinite $\kappa < \lambda$, then T is (κ', λ')-categorical for all infinite $\kappa' < \lambda'$.

Also:

Theorem If T is (κ, κ)-categorical for some uncountable κ,
then for any two models A, B,
every isomorphism from A^P to B^P
extends to an isomorphism from A to B.

This is the one 'abstract' result in the abelian group case known not to be true in general.

S. Shelah and B. Hart, Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$. *Israel J. Math.* 70 (1990) 219–235.

The Kaplansky-Mackey procedure tells us
what information we need about an element a
to extend the isomorphism from a set X to $X \cup \{a\}$.

Hence it isolates the type of a over X.

Thus 'isolated types over a set containing the P-part are dense among types outside the P-part'.

From this point we can call on general model theory.

For example

Theorem If T is (κ, λ)-categorical for some κ and λ,
then for every model E of the P-part T^P of T
there is a model A of T with $A^P = E$.

How do we omit the type of a new element of the P-part?
Answer: In Kaplansky-Mackey we look for
an element c of A^P such that
$a + c$ has maximum height in the coset $a + A^P$.
If $A \preccurlyeq A'$, we won't find a better c in ${A'}^P \setminus A^P$.

A number of questions remain open.

Question One Does every complete first-order theory
that is (κ, λ)-categorical for some κ and λ
have the Reduction Property?

Theories of abelian groups with a distinguished subgroup
behave very like modules.

For example they obey the Baur-Monk quantifier elimination theorem,
and their ω_1-saturated models are classified by
the number of copies of each irreducible pure-injective.

The irreducible pure-injectives are as yet unknown.
Knowledge of them would probably reduce most of the
results above, and the questions below,
to looking up in a catalogue.

Question Two Is it true that if T is a complete theory of pairs of abelian groups, and T^P is a theory of divisible-plus-bounded groups (i.e. ω-stable), then every model of T^P extends to a model of T?

Example Let p be a prime and let T be the theory of the group A of rational numbers whose denominator doesn't contain p^2, with P picking out $\mathbb{Z}_{(p)}$.
Then $A/A^P = \mathbb{Z}(p)^{(\omega)}$.
But there is no model B of T with $B/B^P = \mathbb{Z}(p)^{((2^\omega)^+)}$, since the only models of T^P are $C \oplus \mathbb{Q}^{(\mu)}$ with $C \subseteq \mathbb{J}_p$.

Question Three Under relative categoricity assumptions, can it happen that T has a worse stability classification than both $T \restriction L$ and T^P?

Question Four Which finite pairs of abelian groups are relatively categorical?
(Examples show that A and A^P need not have matching direct sum decompositions.)