Four paradigms for logical games

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Four types of logical game:

- Obligationes
- Dialogue games
- Back-and-forth games
- Hintikka (first-order games and game-theoretic semantics)

There are lots of other kinds of logical game.
But Benedikt asked me to compare these four types,
so I will.

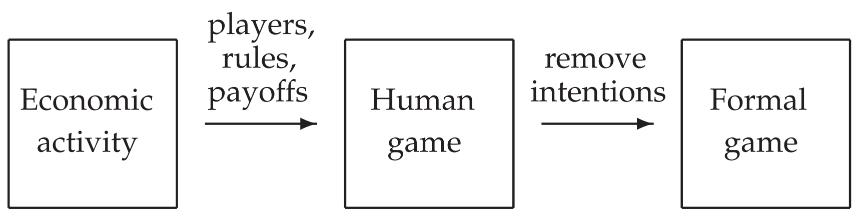
Historical chart:

13–15th cc	Obligationes
1944	Von Neumann and Morgenstern, *Theory of Games and Economic Behavior*
1961	Lorenzen, *Dialogisches Konstruktivitätskriterium*
1961	Ehrenfeucht, *An application of games to the completeness problem for formalized theories*
1964	Hintikka, *John Locke Lectures: Logic, Language, Games and Information*
1983	Hintikka and Kulas, *The Game of Language*

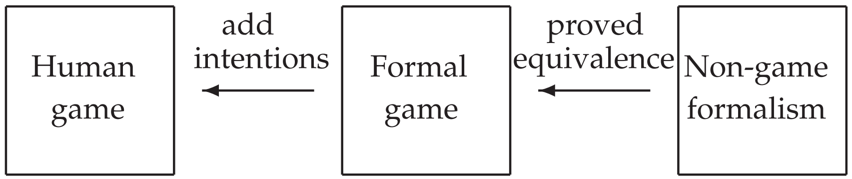
Modelling

When we want to understand a phenomenon P,
we make a *copy* or *model* M of it in terms that we find easier to understand.
To analyse the modelling, we need to identify which features of M represent which features of P.

The Von Neumann-Morgenstern modelling:

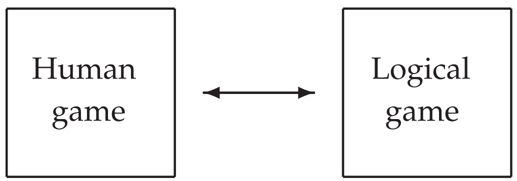


In EF games we will meet another direction of modelling:



At least until recently, logicians using games have been much less careful about modelling than users of games in economics.
The most careful modellers have been in applications of games to evolution (Maynard Smith, Hamilton, Dawkins etc.)

We first review the common part. For our four paradigms the formal game is a *logical game*.



Logical games

A logical game is a pair (G, τ) where

- G is a nonempty set of sequences of length $\leqslant \omega$, which is closed under initial segment and limit;
- $\tau : G \to \{1, 2\}$.

Then G forms a tree branching upwards, under the partial ordering

$$\bar{a} \preccurlyeq \bar{b} \;\Leftrightarrow\; \bar{a} \text{ is an initial segment of } \bar{b}.$$

Maximal elements of G are *plays* of G;
the remaining elements of G are *positions*.

1 and 2 are called *players*.
A play $\bar{a}$ is a *win for* $\tau(\bar{a})$.
A position $\bar{a}$ is a *turn of* $\tau(\bar{a})$.

A *strategy* for player π is a function

$$\Sigma^{\pi} : \text{ (The set of turns of } \pi) \;\to\; G$$

such that for every turn $\bar{a}$ of π,
$\Sigma^{\pi}(\bar{a})$ is immediately above $\bar{a}$ in G.

A play $\bar{a}$ *follows* Σ^{π} if for every turn $\bar{a}{\restriction}n$ of π, $\Sigma^{\pi}(\bar{a}{\restriction}n) \preccurlyeq \bar{a}$.
The strategy Σ^{π} is *winning* if every play that follows Σ^{π} is a win for π.

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The game (G, τ) is *determined* if there exists a winning strategy for one of the players.

We topologise the set of plays by taking as basic open sets the sets of the form

$$\{\bar{a} \in \text{plays} \ : \bar{b} \preccurlyeq \bar{a}\} \quad (\bar{b} \text{ a position}).$$

The *Gale-Stewart Theorem* (1953) says that if for some player π the set of wins for π is open, then the game is determined.

Hence if all plays are finite, the game is determined.

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Logical games count as zero-sum: the *payoff* is a win for one player and a lose for the other.

Suppose Σ^1 and Σ^2 are strategies for players 1, 2.
Then there is a unique play that follows both strategies.
(Hence at least one of the strategies is not winning!)

Von Neumann and Morgenstern give for each game a *strategic* (they say *normalized*) form: the function which, to each pair Σ^1, Σ^2, assigns the payoff of the unique play that follows both strategies. Compared with the strategic form, the original game is said to be in *extensive* form.

At least until recently, logicians have always used the extensive form.

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Strategies define higher positions in terms of lower ones.
So we can read 'higher' as 'later', and imagine two human players 1, 2 creating a play.
Player π names the next position after a turn of π.
A strategy for π is a set of instructions telling π what positions to name.

A winning strategy for π gives π what π needs if π wants to ensure that the play is a win for π.
So we can pass from logical game to human game by supposing that a player with winning strategy *wants to win*.

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Two glitches between human and formal game

(i) going from left to right
Formalising the human game involves choosing a difference between a move that breaks the game rules
and a move that causes the player to lose at once.
This is not always a real difference in the human game.

(ii) going from right to left
If player π has no winning strategy then the idea that π 'wants to win' represents nothing in the formal game.
(This is one reason why the opponent is sometimes blind Nature.)

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Three simplifications of strategies

(1) Since $\Sigma^\pi(\bar{a})$ always has the form

$$\Sigma^\pi(\bar{a}) = \bar{a}^\frown c$$

we can boil Σ^π down to σ^π where $\sigma^\pi(\bar{a}) = c$.

(2) The question whether σ^π is winning depends only on the values $\sigma^\pi(\bar{a})$ where the position $\bar{a}$ 'follows' σ^π.
Define σ_0^π, the *core* of σ^π, to be the restriction of σ^π to these positions.

(3) σ_0^π can be defined as a Skolem function, i.e.
a function of the previous moves of the *other* player.

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A right-to-left modelling: Ehrenfeucht-Fraïssé games

Fraïssé in 1955/6 introduced some formalisms for comparing two structures A, B of the same finite relational signature.

For example we define for each $r < \omega$ a set $G_r(A, B)$ of partial isomorphisms between A and B.
$G_0(A, B)$ is the set of all partial isomorphisms between A and B.
$p \in G_{r+1}(A, B)$ iff: for all a in A there is b in B such that $p \cup \{(a, b)\} \in G_r(A, B)$, and likewise with A, B transposed.

Theorem $A \equiv_r B$ (i.e. they satisfy the same sentences of quantifier rank $\leqslant r$) iff $\emptyset \in G_r(A, B)$.
So $A \equiv B$ iff $\emptyset \in \bigcap_{r<\omega} G_r(A, B)$.

Ehrenfeucht 1961 found a definition of $G_r(A, B)$ through games $EF_\omega(A, B)$.

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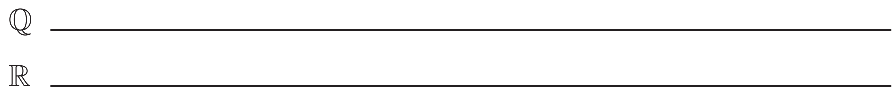
Example where player 2 has a winning strategy



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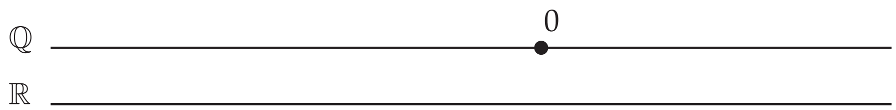

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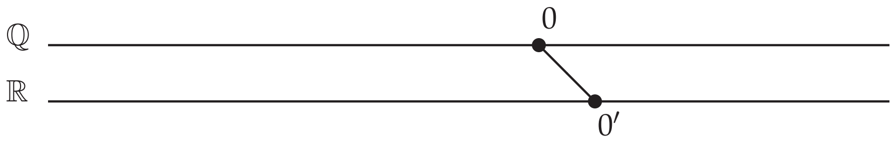

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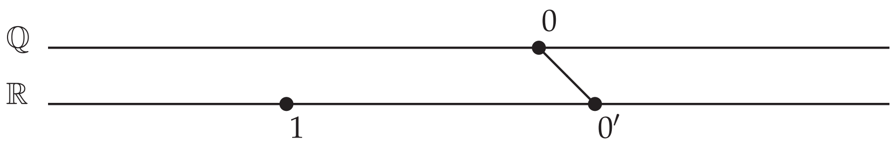

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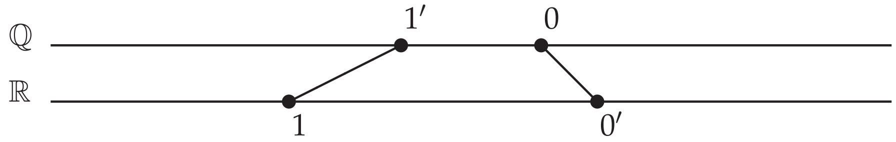

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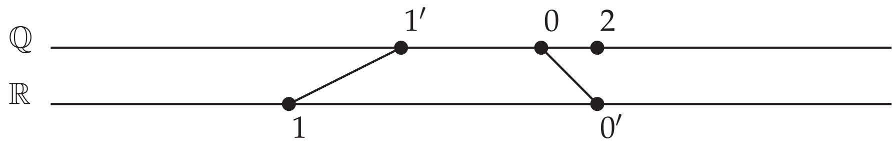

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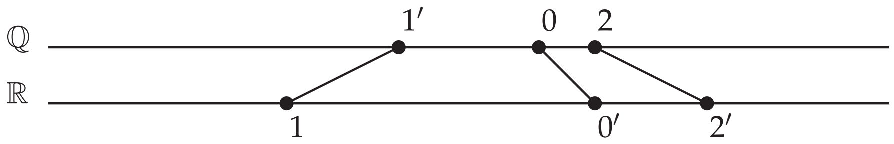

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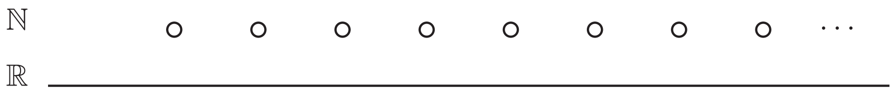
Example where player 1 has a winning strategy



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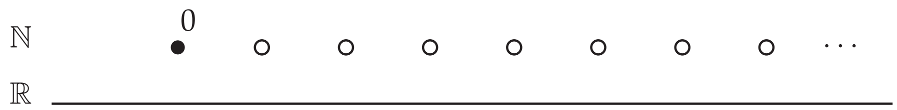

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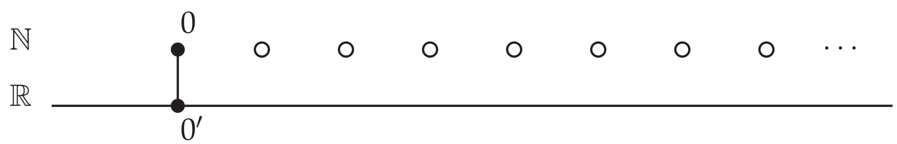

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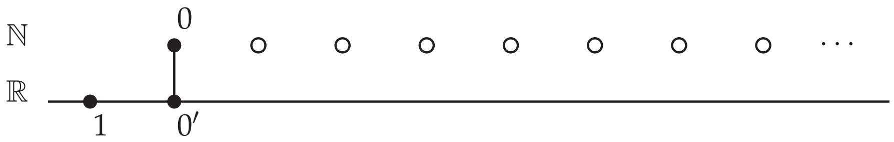

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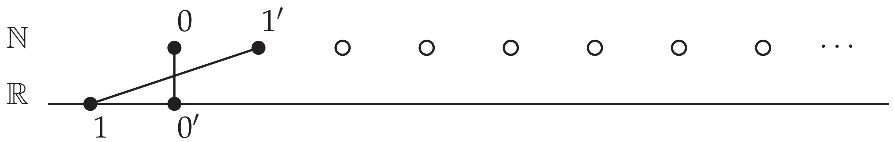

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The second player is today often called *Duplicator*, because this player wins when she can ‘duplicate’ all the first player’s choices.
So a winning strategy for her allows her to achieve the aim of duplicating.

The first player is called *Spoiler*, because a winning strategy for him allows him to spoil Duplicator’s chances of duplicating.

Avoid saying that Duplicator is ‘trying to prove $A \equiv B$’.
A winning play for Duplicator proves nothing at all.
If you are trying to prove $A \equiv B$, you don’t play the game, you *find a winning strategy for Duplicator in the game*.

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A left-to-right modelling: Obligationes

Players are Opponent and Respondent.
Opponent has first move, and from then on the turns alternate between the players.

The activity seems to have been a regular part of Scholastic teaching between the mid 13th century and the early 15th century.
I believe no transcript of a play survives.
But here are two possible reconstructions based on indications in the texts.

Example 1: Respondent wins (based on Burley)

(Arbiter) Obligation: The opponent is not wearing a silly hat.
(Opponent, wearing a silly hat) I propose: Some person is wearing a silly hat.
(Respondent) I respond: True.
(O) I ask: Who is wearing a silly hat?
(R) I respond: I don't know.
(O) I challenge: You do know.
(R) I defend: If I say 'the opponent', I grant a proposition incompatible with the obligation. If I say 'Fred' or 'Cressida' or some other person, I grant a proposition that is false and irrelevant.
(A) The respondent answered well.

Example 2: Opponent wins (based on Heytesbury via Stump)

(A) Obligation: Respondent believes the king is in London.
(O) I propose: You believe that the king is in London.
(R) I respond: True.
(O) I propose: The king is in London.
(R) I respond: I don't know.
(O) I propose: You know that the king is in London.
(R) I respond: I don't know.
(O) I challenge: It is false and irrelevant.
(R) I defend: I can't deny that I know the king is in London without contradicting my previous answers.
(A) The respondent answered badly; the opponent's challenge was good.

The very full account in Paul of Venice *Logica Magna* deduces the rules from assumptions about the general purpose of the activity.

"The topic of *obligationes* is nothing other than the topic of inferences presented in a more subtle manner, in a way intended to test whether the respondent has a good head (for logic) by setting a deceptive course before him. . . . The respondent is taught to keep up his side of the argument steadily and without error." (p. 33)

"Unless this [rule applies], I really do not see how the great logicians, philosophers, geometers and theologians who have adopted a similar way of talking, would be able to speak truly in the cases and assumptions they put before us." (p. 99)

So Paul of Venice is himself modelling the obligatio activity as a human game.

Different authors give different rules for obligationes.
If these authors are all modelling rational inference, then such differences are presumably differences of opinion about rational inference, not just differences in the games.

Confused modelling: Lorenzen's games

Lorenzen described, for each first-order sentence ϕ, a logical game between Proponent and Opponent; Proponent has a winning strategy iff ϕ is intuitionistically provable. In fact a winning strategy for Proponent forms a sort of proof of ϕ.

Some of his rules have no motivation except to ensure that Proponent has a winning strategy for the right sentences. So the modelling is right-to-left, starting with some kind of proof system and modelling it through logical games.

A play that Proponent wins is in fact a part of a proof of ϕ. Opponent has the job of deciding which part. No sensible motivation can be given for Opponent.

So the modelling fails at the stage of human game.

In more detail:

Example 1. Proponent claimed $\psi \wedge \chi$. Then Opponent chooses which of ψ and χ to proceed with. This is a convenience for Proponent, who would otherwise have to prove both ψ and χ.

Example 2. Proponent claimed $\psi \to \chi$. Then Opponent states ψ. But a standard way to prove $\psi \to \chi$ is to state ψ as an assumption and deduce χ; so Proponent would have stated ψ anyway.

Does the separation of players clarify the difference between claims to be proved and assumptions?
No. In answer to $((\psi \to \chi) \to \theta)$, Opponent states $\psi \to \chi$, and then Proponent replies by stating ψ.
So either player can state assumptions.

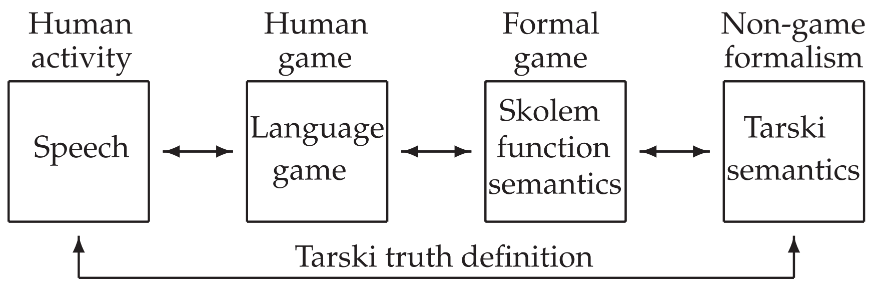
Lorenzen, and some of his defenders, have thought that one could also start from rational debate and reach the same games, thus giving a new 'foundation' for logical validity.

All attempts I've seen fail, through ludicrous assumptions about the players' motivations.

Debit: Lorenzen's games are a paradigm example of obfuscation.

Credit: Some adaptations have been successful for other purposes. E.g. Hyland-Ong computational games, or the games of Krabbe et al. to model rational dialogue.

Complicated modelling: Hintikka semantic games



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Begin at the righthand end, in a formal language for arithmetic:

$$\forall x \exists y \forall z \; (y > x \wedge (y \neq 2z))$$

Skolemise:

$$\exists F \; \forall x \forall z \; (F(x) > x \wedge F(x) \neq 2z)$$

Game: player $\forall$ chooses number a for x, then player $\exists$ chooses b for y. Then $\forall$ chooses c for z, and either left or right conjunct. Player $\exists$ wins iff

$$b > a \quad (\text{resp. } b \neq 2c)$$

If (as here) the sentence is true, a Skolem function F exists. This Skolem function can be read as a winning strategy for $\exists$.

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This generalises to all first-order sentences interpreted in a structure.
The sentence is true iff player $\exists$ has a winning strategy.
If it's true, Skolem functions provide a winning strategy for $\exists$.

Game-Theoretic Semantics generalised this to a wide range of natural language sentences, suitably formalised.

The player $\forall$ was assumed to be Nature.
This is sound, as Hintikka's modelling usually is. But:

(1) He sometimes ascribed motives to Nature. (More recently he has been more cautious about this.)
(2) Against his own better judgement, he sometimes wrote as if a real-life speaker is playing a game against Nature.

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In the more recent IF (Independence-Friendly) logic, Hintikka assumes the Skolem function semantics but allows the sentence to specify that some arguments are missing from the Skolem functions.

$$\forall x \; ((\exists y/\forall x) y = x \vee (\exists y/\forall x) y \neq x)$$

$$\exists c \exists d \; \forall x \; (c = x \vee d \neq x)$$

At $\vee$, player $\exists$ chooses left if $\forall$ chose 0 for x, right otherwise. Then she takes c or d to be 0, independent of the choice for x.

This develops a non-game-theoretic idea of Henkin.
Game-theoretically it's equivalent to requiring that the player chooses with *incomplete information* about the current position.

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Hintikka claimed that in IF logic the modelling has to start with Skolem functions, not with a Tarski-style semantics.

Hodges (1996) gave a Tarski-style semantics for IF logic, equivalent on sentences to the Skolem function semantics.

Väänänen's *Dependence Logic* is an improvement of Hodges' semantics,
and is almost certainly a better starting-point for the IF enterprise than the Henkin-Hintikka Skolem functions.