Relatively categorical abelian groups III

Wilfrid Hodges
Herons Brook, Sticklepath, Okehampton
April 2010
http://wilfridhodges.co.uk


```
''Date:  Sat, 12 Aug 2000 20:53:07
From:  Serban Basarab ⟨Serban.Basarab@imar.ro⟩
To:  Wilfrid Hodges ⟨w.hodges@qmw.ac.uk⟩
Subject:  Re:  exchanges with Royal Society
      Dear Wilfrid,
      Many thanks for your message I just read few
minutes ago.I just arrived from Constanta-the old
Tomis,the exile place of Publius Ovidius Naso-where
I participated to a nice conference on
Algebra-Representations.  ...
Looking forward to meet you, with best wishes,
Yours, Serban''
```

My first introduction to Şerban's work was his fundamental paper in *Journal of Algebra* (1978) on the model-theoretic classification of Henselian valued fields.

Haim Gaifman in *Proceedings of the Tarski Symposium* 1974 (paraphrased):

L a first-order language,
$L(P)$ the result of adding a 1-ary relation symbol P to L.
T a complete theory in $L(P)$, such that in any model A of T the reduct $A{\restriction}L$ to L has a substructure A^P whose elements are those satisfying P in A.
We write T^P for the theory of A^P.

We say T is *relatively categorical* if whenever A and C are models of T, and $i : A^P \to C^P$ is an isomorphism, then i extends to an isomorphism from A to C.

Gaifman's Conjecture: If T is relatively categorical then for every model B of T^P there is a model A of T with $B = A^P$.

Gaifman proved this when L is countable and A is rigid over A^P.

Hodges 1975 showed that Gaifman's argument won't adapt to the case where A is not rigid over A^P.
This is because the natural surjective homomorphism

$$\nu : \mathrm{Aut}(A) \to \mathrm{Aut}(A^P)$$

is not always a split surjection. **Example 1** (p any prime):

$$A = \bigoplus_{i<\omega} \mathbb{Z}(p^2), \quad A^P = pA$$

Gaifman's conjecture has a positive answer when T is countable ω_1-categorical and P picks out a strongly minimal set.

This includes the case of Example 1.
The ω_1-categoricity provides extra structure, as studied by Zilber, Ziegler and Ahlbrandt, Evans etc.

Thanks to work of Zilber and Hrushovski, Example 1 now seems a paradigm example for ω_1-categorical theories with modular geometry.

Saharon Shelah, 'Classification over a predicate II', in *Around Classification Theory of Models*, Springer 1986, pp. 47–90.

Shelah and Leo Harrington both told me that Shelah proved Gaifman's conjecture in this paper.

But Gaifman's conjecture is not stated in the paper.
The paper contains difficult techniques which might yield a proof of Gaifman's conjecture, but (as far as I know) nobody has succeeded in extracting such a proof yet.

Another approach: Get some familiarity with concrete cases.

Wilfrid Hodges, 'Relative categoricity in linear orderings', in *Logic and Algebra*, ed. Yi Zhang, AMS 2002, pp. 235–248. (Models of T are linear orderings with distinguished subordering.)

Wilfrid Hodges and Anatoly Yakovlev, 'Relative categoricity in abelian groups II', *Annals of Pure and Applied Logic* 158 (2009) 203–231. (Models of T are 'group pairs', i.e. abelian groups with distinguished subgroup.)

In both cases Gaifman's conjecture is confirmed.
I am reporting this here for the second case.

The key notion has to be some kind of isolation of types over A^P.
Shelah introduces a rank function for this;
I don't yet understand it.

In the abelian group case, let A be a big model of T, $\bar{a}$ a tuple in A and X a set with $A^P \subseteq X \subseteq A$.
A *support* of $\mathrm{tp}(\bar{a}/X)$ is a formula $\theta(\bar{x}, \bar{y})$ of L such that

(a) for some $\bar{b}$ in X, $A \models \theta(\bar{a}, \bar{b})$;

(b) if $\bar{b}'$ is in X and $A \models \theta(\bar{a}, \bar{b}')$, then $\mathrm{tp}(\bar{a}/X)$ is generated by $\theta(\bar{x}, \bar{b}')$ and $\mathrm{Th}(A, X)$.

Say that a type is *strongly isolated* if it has a support.

We say the relatively categorical theory T satisfies $(\star)$ if:

For every model A of T, if $A^P \subseteq X \subseteq A$ then strongly isolated types are dense among types over X.

We will also use the Reduction Property proved for relatively categorical theories T by Pillay and Shelah 1985:

For every formula $\psi(\bar{y})$ of $L(P)$ there is a formula $(\psi(\bar{y}))^\star$ of L such that for all models A of T,

$$A \models \psi(\bar{b}) \quad \textit{with } \bar{b} \in A^P$$

can be written

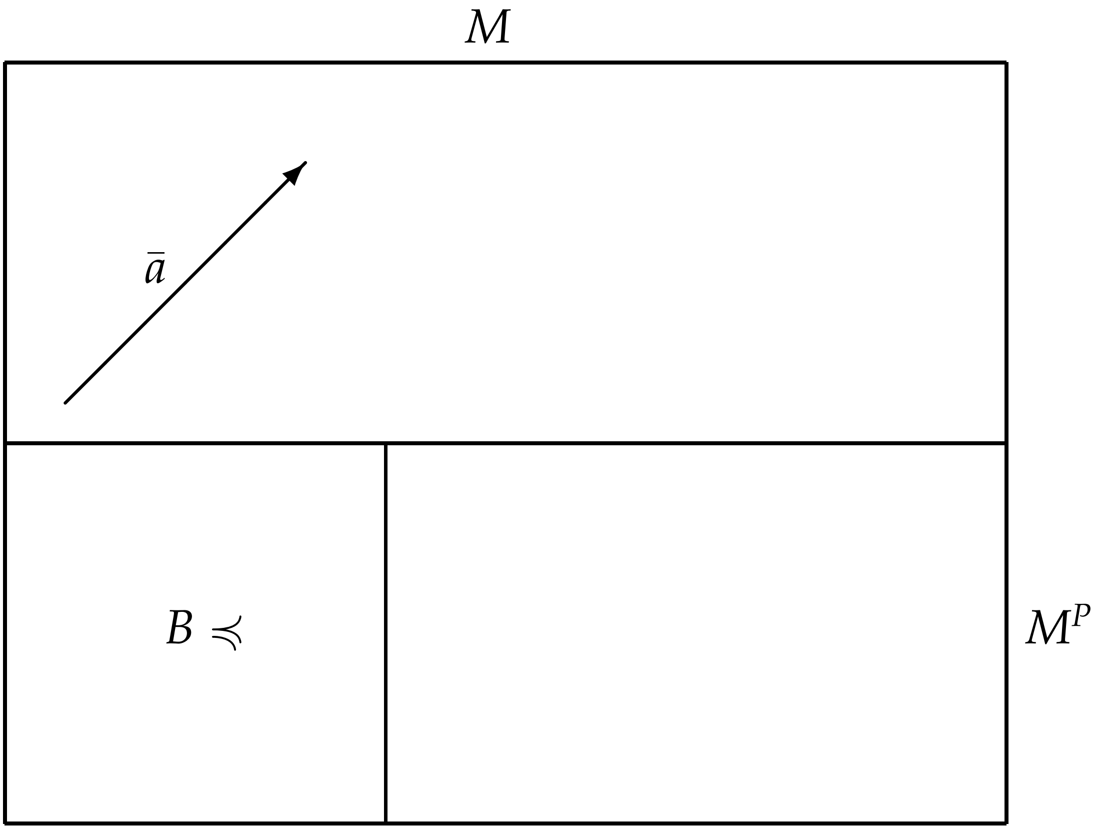
$$A^P \models (\psi(\bar{b}))^\star.$$

Theorem. If T is a relatively categorical theory satisfying $(\star)$ then T satisfies Gaifman's conjecture.

Proof sketch. Take M a big model of T, B a model of T^P. We can assume $B \preccurlyeq M^P$ (in language L).

We inductively build transfinite $\bar{a}$ so that if $M \models \exists x \phi(x, \bar{a})$ then $M \models \phi(c, \bar{a})$ for some c in $\bar{a}$.
Then taking A the set of elements in $\bar{a} \cup B$,
we will get $A \models T$ (by above and Tarski-Vaught),
and $B = A^P$.

11



12

Inductive hypothesis: For each subtuple $\bar{c}$ of $\bar{a}$ there is a support $\theta(\bar{x}, \bar{y})$ with $M \models \theta(\bar{c}, \bar{b})$ for some $\bar{b}$ in B.

A listing (ϕ_i) of formulas is defined in advance.
When $\bar{a} \restriction i$ has been chosen, if $M \models \exists x \phi_i(x, \bar{d})$ with $\bar{d}$ in $(\bar{a} \restriction i) \cup B$ then we will find a_i so that $M \models \phi_i(a_i, \bar{d})$.

By $(\star)$ there is a strongly isolated type over $(\bar{a} \restriction i) \cup M^P$ which includes $\phi_i(x, \bar{d})$, isolated say by $\theta(a, \bar{c}, \bar{d}, \bar{d}')$ with $\bar{c}$ in $\bar{a} \restriction i$, $\bar{d}$ in B and $\bar{d}'$ in $M^P \setminus B$.

Using the inductive hypothesis we can combine a and $\bar{c}$ as a single isolated tuple $\tilde{a}$.
This requires adjusting θ and expanding $\bar{b}$ and $\bar{b}'$.

Key steps:

$$M \models \exists \tilde{x}\theta(\tilde{x}, \bar{b}, \bar{b}')$$

so by the Reduction Property

$$M^P \models (\exists \tilde{x}\theta(\tilde{x}, \bar{b}, \bar{b}'))^\star.$$

Now $B \preccurlyeq M^P$, so for some $\bar{b}''$ in B,

$$M^P \models (\exists \tilde{x}\theta(\tilde{x}, \bar{b}, \bar{b}''))^\star$$

and hence

$$M \models \exists \tilde{x}\theta(\tilde{x}, \bar{b}, \bar{b}'').$$

Since $M \models \exists \tilde{x}\theta(\tilde{x}, \bar{b}, \bar{b}'')$, there is a tuple $\tilde{a}'$ in M with

$$M \models \theta(\tilde{a}', \bar{b}, \bar{b}'').$$

By isolation of $\mathrm{tp}(\tilde{a}'/B)$, we can apply an automorphism of M that fixes B pointwise and takes $\tilde{a}'$ to $a_i{}^\frown\bar{c}$ for some element a_i. Then restoring the original θ we have

$$M \models \theta(a_i, \bar{c}, \bar{d}, \bar{d}'').$$

This finds a_i as required. □

In the abelian group case it was natural to generalise to (κ, λ)-categoricity. Namely T is (κ, λ)-categorical if

1. T has models A of cardinality λ with A^P of cardinality κ, and
2. whenever A, B are models of T with these cardinalities, then every isomorphism $A^P \to B^P$ extends to an isomorphism $A \to B$.

Hodges and Yakovlev 2009 list the possible spectra of relative categoricity for abelian group pairs.
When κ is infinite, there are four:

(1) T is (κ, λ)-categorical just when $\omega \leqslant \kappa = \lambda$.

(2) ... just when $\omega \leqslant \kappa < \lambda$ or $\omega = \kappa = \lambda$.

(3) ... just when $\omega = \kappa = \lambda$.

(4) ... just when $\omega \leqslant \kappa < \lambda$.

Case (1) is the relatively categorical one.

In each case, every model A has the form $C \oplus^P D$ (direct product of $L(P)$-structures) where $A^P \subseteq C$ and C is 'tight' over A^P.

Moreover C is a pushout over A^P of group pairs A_p (p prime) with A_p/A^P a p-group.

'Tight' can be defined several equivalent ways, e.g. (Villemaire 1990) that the Ulm-Kaplansky invariants of each A_p over A^P are all zero.

This allows us to construct each A_p over A^P as in the Kaplansky-Mackey proof of Ulm's theorem (cf. Fuchs *Infinite Abelian Groups II* §77).

To build A over $B = A^P$, we first build the A_p's and form their pushout C; then we build D and take $A = C \oplus^P D$.

Building A_p over $B = A^P$ inside a model A by Kaplansky-Mackey, suppose a is an element of A with $pa \in B$ and $a \notin B$.

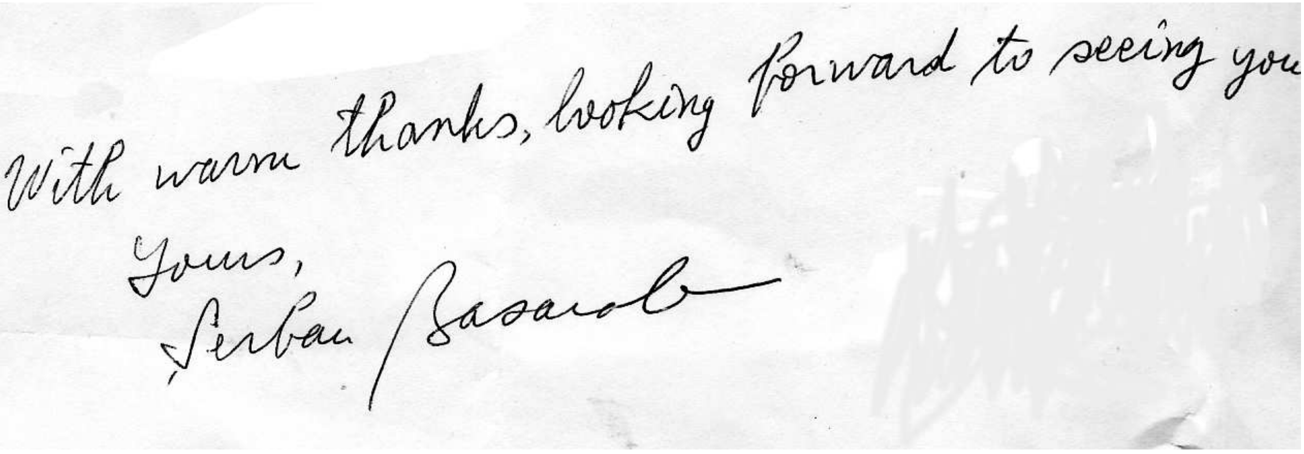
Then K-M choose $b \in B$ so that $a + b$ is an element of maximum p-height in the coset $a + B$.

In the relatively categorical case this p-height will be finite, say k.
We take $\theta(x, y)$ so that $\theta(a, b)$ says: $a + b$ has p-height $\geqslant k$.

This choice of θ as support is the main idea of the proof of $(\star)$ for relatively categorical abelian group pairs.

It remains to ask:

- For what other relatively categorical theories can $(\star)$ be proved?
- Can the proof of Gaifman's conjecture be generalised, e.g. by using partial types as supports?
- Does Gaifman's conjecture hold in full generality for all cases of (κ, λ)-categoricity where the Reduction Property holds? (The generality of the Reduction Property is open.)



And our warmest thanks to you too, Şerban!