Did $\sqrt{2}$ worry the Pythagoreans?

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‘Popular version’

Pythagoras taught that the world is based on rational numbers.

When his followers discovered that $\sqrt{2}$ is irrational, this showed that his main teaching was rubbish.

To prevent public disgrace, they put a curse on anybody who revealed that $\sqrt{2}$ was irrational.

Hippasus leaked the theorem, and died in a shipwreck a few days later.

T. L. Heath, 'A manual of Greek mathematics'

[After conjecturing possible proofs of Pythagoras' theorem ...] 'It must not be overlooked that the Pythagorean theory of proportion was only applicable to commensurable quantities. This would be no obstacle to the use of proportions in such a proof so long as the existence of the incommensurable remained undiscovered. But, when once the incommensurable was discovered, it would be necessary, pending the appearance of a new theory of proportion applicable to incommensurable as well as to commensurable magnitudes, to invent new proofs independent of proportions in place of those in which proportions were used.'

Comment: The historical record contains no evidence whatever of Pythagoreans using the theory of commensurability to prove geometric theorems. No Greek writer ever mentions a crisis of the kind that Heath postulates for the Pythagoreans.

E. R. Dodds
'The Greeks and the irrational'
University of California Press 1951

English

- RATIONAL = in accord with reason
- RATIONAL = fractional

Classical Greek

- LOGOS = explanation, underlying structure
- LOGOS = proportion

Anonymous and undated **Scholium** to Euclid, 'Elements'

'The Pythagoreans were the first to make inquiry into commensurability, having first discovered it as a result of their study of numbers. For though the unit is a common measure of all numbers they could not find a common measure of all magnitudes [since for example any magnitude can be halved . . .]. There is a legend that the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck.'

Iamblichus, 'On the Pythagorean life'

'It is related of Hippasus that he was a Pythagorean, and that, owing to his being the first to publish and describe the sphere from the twelve pentagons, he perished at sea for his impiety, but he received credit for the discovery, though really it all belonged to "that man" (as Pythagoras was known to his followers).'

(**Iamblichus** continued)

'[Hippasus] was the first person who explained the nature of commensurability and incommensurability to the unworthy, and as a result he was excluded from the [Pythagorean] community; in fact they even built a tomb for him as if he was dead.'

'[The Pythagoreans] say that God takes revenge on anyone who makes public the teachings of Pythagoras. In fact someone did die at sea, as if punished for sacrilege, after publishing a method for constructing the icosagon [icosahedron?] and showing how to inscribe the dodecahedron in a sphere. But some say that his punishment was for speaking out about disproportion and incommensurability.'

Four Greek themes:

1. The orderly and the disorderly.
2. Secrecy.
3. The physical facts of music.
4. The psychological facts of music.

1. The orderly and the disorderly

Conservative	**Freaked out**
Is orderly	Is disorderly
Is moderate	Goes to extremes
Doesn't	Does have bizarre eating habits
Doesn't	Does use mind-altering substances
Doesn't	Does fly, visit Hades
Institutional religion	Direct line to God
Admires Tony Blair	Admires Will Self

Aristotle, 'Metaphysics'

'Some of [the Pythagoreans] say there are ten principles, which they arrange in two columns:

Limit	Unlimited
Odd	Even
One	Plurality
Right	Left
Male	Female
Resting	Moving
Straight	Curved
Light	Darkness
Good	Bad
Square	Oblong'

2. Secrecy

The chief Greek mystery (thanks to its sponsorship by Athens) was the mystery of Demeter at Eleusis.
The idea was to undergo a ritual lasting several days.
The final stage was a 'spectacle'.
After seeing the spectacle, one became a 'beholder' (i.e. initiate) and was sworn to secrecy about the spectacle.
By tradition the rites came from Demeter herself, and they were

> Holy things which one cannot divulge or utter;
> A great awe of the gods holds back one's voice.

Compare:
Freemason rituals
The Mousetrap.

Iamblichus:

‘The strictness of [Pythagorean] secrecy is astonishing; for in so many generations evidently nobody ever encountered any Pythagorean notes before the time of Philolaus.’

Porphyrius, ‘Life of Pythagoras’:

‘What he said to his associates, nobody can say for certain; for silence with them was of no ordinary kind.’

Other restrictions:

Never eat beans or red mullet.
Don’t let swallows nest under your roof.
Never stir the fire with a knife.
Always make your bed.
Never laugh uncontrollably.
Always wash your left foot before your right.
Never pee in the direction of the sun, or onto your nail clippings.
When you go abroad, don’t change your mind at the frontier.

3. The physical facts of music

A plucked string makes a note.
The note rises by an octave if one stops the string half way;
an octave and a fifth if two-thirds of the way;
two octaves if three-quarters;
two octaves and a third (a discord in Greek music) if four-fifths.

Aristotle, 'Metaphysics'

'[The Pythagoreans] saw that the attributes and the ratios of the musical scales were expressible in numbers.'

Evidence that they stopped at the fourth harmonic (including the tonic).

Kirk and Raven: 'There seems no reason to doubt the tradition that Pythagoras himself discovered . . . that the chief musical intervals are expressible in simple numerical ratios between the first four integers.'

Unfortunately no reason to believe it either.

4. The psychological facts of music

Helmholtz, 'Die Lehre von den Tonempfindungen' (1885):

'The enigma which, about 2500 years ago, Pythagoras proposed to science, which investigates the reasons of things, 'Why is consonance determined by the ratios of small whole numbers?', has been solved ... by Fourier's law.'

This completely misses the point. At Pythagoras' date there was no physics, and consonance of a musical interval was a fact about the *human mind*, not about the physical world.

Boethius: Seeing a young man about to commit arson out of jealousy, Pythagoras noticed that there were musicians playing in the Phrygian mode. He told them to change the mode to something more relaxing, and the young man calmed down at once.

Thus the tradition that Pythagoras invented music therapy.

Questioner:

Dear Mrs Pythagoras,

$$\pi o \sigma \tau \alpha \iota \alpha \ \gamma \upsilon \nu \eta \ \alpha \pi' \ \alpha \nu \delta \rho o s \ \kappa \alpha \theta \alpha \rho \epsilon \upsilon \epsilon \iota ?$$

Mrs Pythagoras replies:

$$\alpha \pi o \ \mu \epsilon \nu \ \tau o \upsilon \ \iota \delta \iota o \upsilon \ \pi \alpha \rho \alpha \chi \rho \eta \mu \alpha ,$$
$$\alpha \pi o \ \delta \epsilon \ \tau o \upsilon \ \alpha \lambda \lambda o \tau \rho \iota o \upsilon \ o \upsilon \delta \epsilon \pi o \tau \epsilon .$$

(This is for real. See Diogenes Laertius viii.42.)

Cross, Halcomb and Matter, 'Imprinting or exposure learning in rats given early auditory stimulation' (1967):

Newborn albino rats were raised in acoustic chambers. One group was exposed to a selection of Mozart's compositions for several weeks. A second group was exposed to Schoenberg's music. Later, when given an opportunity to select either Mozart's or Schoenberg's music in a choice box, the Mozart-trained group preferred Mozart, but the Schoenberg-trained group showed no preference for Schoenberg.

Crucial point: At Pythagoras' date, a numerical explanation of mental phenomena was probably more plausible than one of physical phenomena.
We have to imagine what the world felt like before Galileo and Newton.

Conjectural summary

Pythagoras invents cult based on:

- The usual crazy observances, including secrecy.
- The order/disorder dichotomy as universal principle.
- The causal role of whole numbers, allowing an arithmetical 'explanation' of the existence of order ('cosmos') in the mind, and hence in the whole world.

Pythagorean community attracts people with a genuine interest in numbers and shapes, for example Hippasus. These people contribute to the theories of irrationals and Platonic solids.

Pythagorean community probably regards the irrationality of $\sqrt{2}$ as giving mystic 'explanation' of the existence of disorder.

Conflict of mathematicians with religious-minded Pythagoreans, who object to cult knowledge being made public. Silly stories about this conflict spread, reaching written literature several centuries later.