From sentence meanings to full semantics

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1. Two approaches to meanings

A. Aristotle (4th c. BC)

Separate words have meanings.
The meaning of a sentence is the result of combining the meanings of the words in it.

Georg Friedrich Meier (1757)

> The meaning of a sentence is the embodiment of all the separate meanings of the words which make up the sentence; they are bound to each other and determine each other.

B. Abdul Qāhir al-Jurjānī (Persia, 11th c)

Meanings are expressed primarily in sentences.
Individual words have meanings,
but the meaning of a word is not detachable from the roles it plays in sentences.

Dalāil p. 316

> When you say ‘Zaid beat up Amran last Friday’, you present not the meanings of the separate words but the connections between these meanings in the sentence.

Let L be a language (natural or artificial).

We assume each word and each sentence of L has a meaning.

Hence two functions

δ : words $\rightarrow$ word-meanings.
σ : sentences $\rightarrow$ sentence-meanings.

The central problem is to relate δ to σ.

How do we work out what this means?

> His Majesty may by Order in Council transfer to, or make exercisable by, the Minister any of the functions of the Charity Commissioners in matters appearing to His Majesty to relate to education, and any such Order may make such provision as appears to His Majesty to be necessary for applying to the exercise of those functions by the Minister any enactments relating to the Charity Commissioners.

Sentences break down into *constituents*.

We can separate out

(a) a constituent C of a sentence S,
(b) the rest of S when C is removed.

The rest of the sentence is a *frame*,
i.e. an expression with a variable, that becomes a sentence when the variable is replaced by a suitable expression.

This is recursive;
we can also separate out constituents of constituents.

Definition. By a *constituent structure* we mean an ordered pair of sets $(\mathbb{E}, \mathbb{F})$, where the elements of $\mathbb{E}$ are called the *expressions* and the elements of $\mathbb{F}$ are called the *frames*, such that the four conditions below hold.

(e, f etc. are expressions. F, $G(\xi)$ etc. are frames.)

1. $\mathbb{F}$ is a set of nonempty partial functions on $\mathbb{E}$.

('Nonempty' means their domains are not empty.)

2. (Nonempty Composition) If $F(\xi_1, \ldots, \xi_n)$ and $G(\eta_1, \ldots, \eta_m)$ are frames, $1 \leqslant i \leqslant n$ and there is an expression

$$F(e_1, \ldots, e_{i-1}, G(f_1, \ldots, f_m), e_{i+1}, \ldots, e_n),$$

then

$$F(\xi_1, \ldots, \xi_{i-1}, G(\eta_1, \ldots, \eta_m), \xi_{i+1}, \ldots, \xi_n)$$

is a frame.

Note: If $H(\xi)$ is $F(G(\xi))$ then the existence of an expression $H(f)$ implies the existence of an expression $G(f)$.

3. (Nonempty Substitution) If $F(e_1, \ldots, e_n)$ is an expression, $n > 1$ and $1 \leqslant i \leqslant n$, then

$$F(\xi_1, \ldots, \xi_{i-1}, e_i, \xi_{i+1}, \ldots, \xi_n)$$

is a frame.

4. (Identity) There is a frame $1(\xi)$ such that for each expression e, $1(e) = e$.

We say e is a *constituent* of f if f is $G(e)$ for some frame G.

$F(e_1, f, e_3)$ is *the result of replacing the occurrence of* e_2 *in second place in* $F(e_1, e_2, e_3)$ *by* f.
(This notion depends on F, of course.)

Every bare grammar in the sense of Keenan and Stabler, *Bare Grammar*, CSLI 2003, has a constituent structure in an obvious way.

2. The lifting lemma

Let X be a set of expressions (for example the sentences) and $\mu : X \to Y$ any function (for example σ).

We will define a relation $\sim_\mu$ so that

$$e \sim_\mu f$$

says that expressions e and f make the same contribution to μ-values of expressions in X.

The fact that $\sim_\mu$ must be an equivalence relation more or less forces us to the following definition.

Definition We write $e \sim_\mu f$ if for every 1-ary frame $G(\xi)$,

- $G(e)$ is in X if and only $G(f)$ is in X;
- if $G(e)$ is in X then $\mu(G(e)) = \mu(G(f))$.

We say e, f have *the same* $\sim_\mu$*-value*, or for short *the same fregean value*, if $e \sim_\mu f$.
We write $|e|_\mu$ for this fregean value
(determined only up to $\sim_\mu$).

LEMMA. Suppose $F(e_1, \dots, e_n)$ is a constituent of some expression in X, and for each i, $e_i \sim_\mu f_i$. Then:

(a) $F(f_1, \dots, f_n)$ is an expression.

(b) $F(e_1, \dots, e_n) \sim_\mu F(f_1, \dots, f_n)$.

For the **proof**, by Nonempty Substitution we can make the replacements one expression at a time.
So it suffices to prove the lemma when $n = 1$.

Assume $F(e)$ is an expression, $H(F(e))$ is in X
and $e \sim_\mu f$.

Proof that $F(f)$ is an expression.
By Nonempty Composition $H(F(\xi))$ is a frame $G(\xi)$.
Since $e \sim_\mu f$ and $G(e)$ is in X, $G(f)$ is in X.
But $G(f)$ is $H(F(f))$, so $F(f)$ is an expression.

Proof that $F(e) \sim_\mu F(f)$.
Let $G(\xi)$ be any 1-ary frame such that $G(F(e))$ is an expression in X.
By Nonempty Composition $G(F(\xi))$ is a frame $J(\xi)$.
Since $e \sim_\mu f$ and $J(e)$ is in X,
$J(f)$ is in X and $\mu(J(e)) = \mu(J(f))$.
So $\mu(G(F(e)) = \mu(G(F(f))$ as required. □

We say that X is *cofinal* if every expression is
a constituent of an expression in X.

Basic example:

- L is a language,
- $(\mathbb{E}, \mathbb{F})$ is the constituent structure of L,
- X is the set of sentences of L,
- for each sentence e, $\mu(e)$ is the class of contexts in which e is true.

Assume X is cofinal. Then by (b) of the Lemma,
if $e_i \sim_\mu f_i$ for each i then $F(e_1, \ldots, e_n) \sim_\mu F(f_1, \ldots, f_n)$
provided these expressions exist.
So F and the fregean values of the e_i
determine the fregean value of $F(e_1, \ldots, e_n)$.

Hence there is, for each n-ary frame F, an n-ary map
$h_F : V^n \to V$, where V is the class of $\sim_\mu$-values,
such that whenever $F(e_1, \ldots, e_n)$ is an expression,

$$|F(e_1, \ldots, e_n)|_\mu = h_F(|e_1|_\mu, \ldots, |e_n|_\mu).$$

$$|F(e_1, \ldots, e_n)|_\mu = h_F(|e_1|_\mu, \ldots, |e_n|_\mu).$$

We call h_F the *Hayyan function* of F.

Abu Ḥayyān al-Andalusī (Egypt, 14th c.)
argued that such functions must exist,
from the fact that we can create and use new sentences.

Using the Hayyan functions,
the values $|e|_\mu$ are definable by recursion on the syntax,
starting from $|f|_\mu$ for the smallest constituents f.

Definition Let ϕ be a function defined on expressions.
A definition of ϕ is called *compositional* if
for each expression $F(e_1, \ldots, e_n)$,

$$\phi(F(e_1, \ldots, e_n))$$

is determined by F and the values $\phi(e_i)$.

So fregean values are compositional.

We have derived Aristotle's viewpoint from Jurjānī's.

One can construct counterexamples to the converse implication:
there are compositional semantics that don't yield fregean values,
because they carry up redundant information.

E.g. game-theoretic semantics, where two different games can correspond to the same fregean value.

3. When is the lifting an extension?

PROPOSITION. Suppose $e \sim_\mu f$ and e is an expression in X. Then f is in X and $\mu(e) = \mu(f)$.

Proof. This is immediate from the definition,
by applying the identity frame $1(\xi)$. $\square$

So on X the relation $\sim_\mu$ is a refinement of the relation $\mu(\xi_1) = \mu(\xi_2)$.

This guarantees that the information carried up through $\sim_\mu$ is enough to determine μ on expressions in X.

PROPOSITION. The relation $\sim_\mu$ extends the relation $\mu(\xi_1) = \mu(\xi_2)$ if and only if:

For all e, f in X and every frame $F(\eta)$,

$$\begin{aligned} & \mu(e) = \mu(f) \text{ and } F(e) \in X \\ \Rightarrow \quad & F(y) \in X \text{ and } \mu(F(e)) = \mu(F(f)). \end{aligned}$$

Proof again immediate from the definition. $\square$

These assumptions on μ are rather restrictive.
A major problem is to prove the existence of (weaker) extensions of μ under weaker hypotheses.

Dag Westerståhl proves: If

- μ is compositional on a set X of expressions of L,
- domain of μ is closed under constituents,
- L is a subset of a term algebra,

then μ can be extended to a compositional function on L.
The hard part is to circumvent the failure of

$$\mu(e) = \mu(f) \text{ and } F(e) \in X \;\Rightarrow\; F(f) \in X.$$

4. Applications to artificial languages

Since Tarski 1933, the semantics of languages of logic is in terms of

> necessary and sufficient conditions for a sentence ϕ to be true under a given interpretation.

Define

$$\mu(\phi) = \{A : A \text{ an interpretation of } L \text{ which makes } \phi \text{ true}\}.$$

This function is normally compositional on the set X of sentences.

By the lifting lemma there is a compositional function ν defined on all the expressions of L, such that if ϕ and ψ are in X and $\nu(\phi) = \nu(\psi)$ then $\mu(\phi) = \mu(\psi)$.

To build up necessary and sufficient conditions for a sentence to be true, we need only

- define $\nu(e)$ for each atomic expression e,
- define the Hayyan function of each frame F,
- define the set of ν-values of true sentences.

In practice the conditions for the fregean cover function ν to be an extension of μ are nearly always met, and the values $\nu(e)$ for atomic expressions are the obvious ones.

For example if R is a binary relation symbol, $\nu(R)$ is normally (essentially) the function taking each interpretation A to the set R_A of all ordered pairs satisfying R in A.

The Hayyan functions rarely give problems.

Tarski's truth definitions follow exactly this pattern (though he never said so).

However, there are logical languages that need a more complex definition for the Hayyan functions.

Cameron and Hodges study one and show that fregean values can't be defined in terms of satisfaction.

The language in question (due mainly to Leon Henkin) formalises an idiom used often by Serge Lang:

Serge Lang, *Algebraic Number Theory* (2nd edition) p. 335:

"Let $0 < a \leq 1$, and m an integer with $|m| \geq 2$. Let $s = \sigma + iT_m$ with $-a \leq \sigma \leq 1 + a$ and T_m as above. Then

$$|\xi'/\xi(s)| \leq b(\log|m|)^2,$$

where b is a number depending on a but not on m and σ."

Tarski's semantics would require us to interpret the relation

$$xRy: \quad x \text{ is a number depending on } y.$$

For example is 1984 a number depending on 1776?

Cameron and Hodges characterise the fregean values for this language L and then count them.

E.g. the number of fregean values of first-order formulas with one free variable, interpreted in an 8-element structure, is at most

$$2^8 = 256.$$

For the language L the number is

$$112,260,874,496,010,913,723,317.$$

Moral: Tarski's use of satisfaction is an accidental (but very convenient) feature of first-order logic.

In spite of their complexity, these fregean values are useful.

Julian Bradfield (2003) used them to define a fixed point logic of 'imperfect information'.

Rohit Parikh and Jouko Väänänen (in press) use them to provide a semantics for a 'finite information logic'.

5. Applications to natural languages

In natural languages the words already have meanings, as in the dictionary.

Does 'having the same meaning' coincide with 'having the same fregean value'?
I.e. have we really recovered δ from σ?

In general no, but our setting is a classifiers' paradise.
The usual pattern is that fregean values *refine* δ.

Compare ‘liked’ and ‘enjoyed’:

- I liked / enjoyed playing tennis.
- I liked / enjoyed the chocolate cakes.
- I liked / enjoyed the Beatles song.
- I liked / enjoyed the pencils.
- I liked / enjoyed Mary.

The pattern seems to be that ‘liked’ and ‘enjoyed’
mean the same except that
‘enjoyed’ can only be used of activities.

So some frames accept ‘liked’ but reject ‘enjoyed’.

Pustejovsky gives various examples of this kind.

Compare ‘murderer’ and ‘person who has killed someone’:

> That man is the murderer of Caroline.
> That man is the person who has killed someone of Caroline.

The second sentence is syntactically ugly, but
the main point is that it can’t mean the same as the first.
‘murderer’ has an argument place not available in
‘person who has killed someone’.

In these examples the fregean value seems
a more sensitive indicator of meaning
than a naive dictionary definition would be.

Jurjānī, *Dalāil*:

> ‘Understand that I am certainly not saying that
> the mind doesn’t grasp hold of the meanings
> of separate words.
> What I am saying is that it doesn’t grasp hold of
> the meanings of separate words detached from
> the meaning of their syntax.

Less interesting but still worth puzzling over, compare ‘notwithstanding’ and ‘in spite of’:

> They completed the journey, bandits notwithstanding.
>
> They completed the journey, bandits in spite of.

Marcus Werning has a paper in press arguing that the lifting lemma shows there is much less indeterminacy in translation than Quine claims.

Hannes Leitgeb has written a reply.

6. References

- Julian Bradfield, ‘Parity of Imperfection, or Fixing Independence’, in *CSL ’03*, Lecture Notes in Computer Science 2803, Springer, New York 2003.
- Peter Cameron and Wilfrid Hodges, ‘Some combinatorics of imperfect information’, *Journal of Symbolic Logic* 66 (2001) 673–684.
- Tim Fernando and Dag Westerståhl, ESSLLI 2001 lecture notes at `www.helsinki.fi/esslli/courses/CaC.html`
- Wilfrid Hodges, ‘Formal features of compositionality’, *Journal of Logic, Language and Information* 10 (2001) 7–28.

- Wilfrid Hodges, ‘What languages have Tarski truth definitions?’, *Annals of Pure and Applied Logic* 126 (2004) 93–113.
- Rohit Parikh and Jouko Väänänen, ‘Finite information logic’, *Annals of Pure and Applied Logic* to appear.
- Jouko Väänänen, ‘On the semantics of informational independence’, *Logic Journal of the IGPL* 10 (2002) 337–350.
- Dag Westerståhl, ‘On the compositional extension problem’, *Journal of Philosophical Logic* to appear.