Necessity in mathematics

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Abstract

Mathematical writing commonly uses a good deal of modal language. This needs an explanation, because the mathematical assumptions and arguments themselves normally have no modal content at all. We review the modal expressions in the first hundred pages of a well-known algebra textbook, and find two uses for the modal language there: (a) metaphors of human powers, used for colouring that helps the readability; (b) formatting expressions which highlight the structure of the reasoning. We note some ways in which historians and philosophers of mathematics might have been misled through taking these modal expressions to be part of the mathematical content.

Facts A and B below seem at first sight to be inconsistent with each other. So we have a paradox.

FACT A: Mathematics contains no modal notions.

For example one sufficient condition for the correctness of a mathematical argument is that it should be formalisable as a proof in Zermelo-Fraenkel set theory. Zermelo-Fraenkel set theory has just two primitive notions, 'set' and 'is a member of'. Neither of these notions is modal.

Of course mathematics is full of necessary truths, for example this theorem of analysis:

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$$

But only philosophers are interested in the fact that this theorem is *necessarily* true. Mathematicians are content if they can show that it is true.

FACT B. Mathematical writing is full of modal notions.

You can check this by glancing at almost any mathematical text. To be more objective I took the first hundred pages of a well-known textbook, Birkhoff and Mac Lane's *A Survey of Modern Algebra* [1], and listed all the instances of modality. I included for example 'allow', 'can', 'cannot', 'could, 'essential', 'have to', 'impossible', 'inevitably', 'may', 'might', 'must', 'necessarily', 'need not', 'need only', 'possibility', 'possible', 'will'. I found 340 examples, or 3.4 examples per page.

The rest of this paper will try to resolve the paradox. The main work is to examine in detail some of the examples from Birkhoff and Mac Lane. If the modal expressions are not there to express any modality in the mathematics, they must be there for some other reason or reasons. I hope to show at least broadly what these reasons are. The list will be referred to as BM, and items from BM will be marked with a reference ($m.n$) meaning line n on page m.

I chose Birkhoff and Mac Lane [1] for two main reasons. The first is that the mathematics involved is clean and self-contained. The second reason is that this book had in its heyday a well-deserved reputation for clear writing and arrangement. But I have to add a caution: the book is now over fifty years old, and the use of modal terms in English has shifted perceptibly since then. For example modal 'may' has dropped out of use except in highly formal contexts, and 'can' has largely replaced it. Likewise 'must' is yielding to 'needs to' and 'has to'. There are also some differences between the two sides of the Atlantic. See Facchinetti et al. [3], and particularly the essays in it by Leech [4] and Smith [6].

Within the present limits of time and space I can only explore a part of what BM tells us. It seemed best to probe a few of the themes in depth rather than present a shallow catalogue of the whole. I hope to be able to give a fuller analysis sometime soon. Meanwhile I warmly thank the Philosophy Department of Jadavpur University, and in particular Professors Prabal Kumar Sen and Dilipkumar Mohanta and their helpers, for organising the stimulating conference where this paper was presented.

1 Modality inside theorems

Of the 340 items in BM, 41 (=12%) occur inside theorems, definitions or exercises, as part of the mathematical content.

Here are three of them:

(Theorem) (53.8) The system J^+ **can be** embedded in a larger system in which subtraction is **possible**.

(Definition) (90.15) By an upper bound to a set S of elements of an ordered domain D is meant an element b (which **need not** itself be in S) such that etc.

(Exercise) (67.22) Show that one **cannot** embed in a field the ring $J_p\langle x\rangle \dots$

All these 41 cases must have paraphrases not using modal terms, and it's assumed that the reader can work out the paraphrases. Otherwise the theorems aren't proved, the definitions are unusable and the exercises are unsolvable.

For example in (53.8) above, the phrase 'subtraction is possible' does have a straightforward mathematical paraphrase, and if we replace the phrase by its paraphrase we get

(1) The system J^+ can be embedded in a larger system in which for all elements x and y there exists an element z such that $x + z = y$.

Comparing (1) with (53.8), we can see an obvious reason to prefer (53.8). The labour of parsing (1) is a distraction to the reader. Moreover any reader with a certain amount of mathematical maturity will see, at least after reading pages 1 to 52, that (1) does express what is intended by (53.8).

One well-known analysis states that in mathematics the word 'possible' has a non-modal use as a kind of existential quantifier. This seems to me unhelpful. The students were never taught this use; they were expected to pick up (1) from the mathematical context and the normal usage of 'possible'. In fact the ingredients of the phrase

(2) subtraction is possible

don't match the ingredients of

(3) for all elements x and y there exists an element z such that $x + z = y$

item by item. If 'is possible' in (2) corresponds to 'there exists' in (3), then presumably the rest of (3) corresponds to the one word 'subtraction' in (2). This is implausible; (3) doesn't contain any ingredient with the meaning 'subtraction'. A more convincing description would be that the word 'possible' invites the reader to look for certain kinds of mathematical feature in the context and formulate (1) or something equivalent out of them. This gives 'possible' something of the status of a macro.

Is there any prospect of interpreting away all occurrences of modal words like 'possible' in the same way as we did in (1)? This question scratches the surface of a deep question. But on the face of it, the answer is no. The word 'can' does in some cases have a purely descriptive content within a certain range of concepts, and it's not plausible that we can explain this content without using the concepts in question. Consider for example

(4) My cat can't see colours.
Plants with chlorophyll can extract energy from sunlight.

In both these examples, 'can' is part of the vocabulary of organisms. To say 'X can Y' is to say that the organism X has a certain power, namely to do what is expressed by Y. We can hardly explain what it is for an organism to have a power without in some way invoking the notion of an organism. (Some linguists suggest that cases like (4) are metaphorical extensions of the notion of a human power. This wouldn't affect the point being made here.)

A handful of the 340 items in BM do literally refer to human powers:

> (93.10) We can **visualize** the above proof as follows.
> (79.20) We may typically **imagine** G as the domain of the integers ....

These passages both invite the reader to use certain mental powers as a help for understanding the mathematics. They are not really part of the mathematical content. They belong in the same world as the warning 'The reader is advised to sit down before beginning this proof' (Chang and Keisler [2]) p. 101).

2 'Embedded'

The examples (53.8) and (67.22) above use the notions 'can be embedded' and 'cannot embed'. Before we can make sense of the modals here, we need to clear up a problem about the verb 'embed'. The problem is that mathematicians never define what it means. Most mathematical textbooks that use the notion define the noun 'embedding (of Y in Z)' as a certain kind of function, but they don't define the verb that it is supposed to come from. Birkhoff and Mac Lane are quite unusual in defining the adjective 'embedded (in)' ([1] p. 43); but not even they define 'embed'.

It looks at first as if the definition of 'embedded' covers the phrase 'can be embedded' in (53.8). On closer inspection this doesn't work. The phrase 'be embedded' in (53.8) is the passive of the verb 'embed', not 'be' plus an adjective 'embedded'. We can see this from the presence of the modal 'can'. Take an adjective like 'prime'; what would it mean to say that the number 5 'can be prime' (except perhaps that we don't know whether 5 is prime)? Or 'high'; what would it mean to say that Everest 'can be high'? Either 5 is prime or it isn't, and there is no room here for possibility. Likewise in the adjectival sense of 'embedded' that Birkhoff and Mac Lane define, either the system J^+ is embedded in a larger system as in (53.8) or it isn't—end of story.

So (53.8) needs a definition of the verb 'embed'. The schema that needs defining is 'X embeds Y in Z'. There are two possible usages:

(5) (a) Function f embeds system J in system K.
(b) Person A embeds system J in system K.

We saw that 'be embedded' in (54.8) is the passive of the verb 'embed'; but the passive hides the agent and doesn't immediately tell us whether we have the passive of (a) or (b). However, we can eliminate (a) by an argument like the one that showed we have a verb here. The phrase (a) has the same overall form as

(6) The Golden Gate Bridge joins San Francisco to Sausalito.

But in this sense of 'join' we would never say

(7) San Francisco can be joined to Sausalito

The natural reading of (7) is that there is no connecting bridge from San Francisco to Sausalito, but somebody can build one. This and similar examples show that 'be embedded' in (53.8) comes from (b), not from (a). What (53.8) says is that people—mathematicians—can produce a suitable embedding. The 'one' in (67.22) shows that 'embed' here is (b) again.

There is no problem about giving a mathematical definition of (a); it says that f is an embedding of J in K. Then we can explain (b) in terms of (a), for example as saying that person A defines a function f as in (a). The problem now is that with 'can', this gives the wrong meaning in (53.8) and (67.22).

On the reading just given, (67.22) invites the reader to show that there is no available definition of an embedding of $J_p\langle x\rangle$ in a field. But the reader—an algebra student—has been taught nothing about how to show that definitions aren't available. The only way the student could hope to show that we can't give a definition of such an embedding is to show that there is no such embedding. And this is certainly what Birkhoff and Mac Lane wanted the student to show. So why the irrelevance about definitions?

More precisely, why did Birkhoff and Mac Lane write their exercise as in (a) below and not as in (b)?

(8) (a) Show that one cannot embed in a field the ring $J_p\langle x\rangle \ldots$
(b) Show that there is no embedding of the ring $J_p\langle x\rangle \ldots$ in a field.

The length and complexity of (a) are roughly equal to those of (b). The chief difference is the irrelevant suggestion of human powers in (a). So why choose (a)?

Here is the corresponding question for (53.8).

(9) (a) The system J^+ can be embedded in a larger system in which subtraction is possible.
(b) There is an embedding of the system J^+ in a larger system in which subtraction is possible.

Again (b) has roughly the length and complexity as (a). Why did Birkhoff and Mac Lane write their theorem as (a) and not as (b)?

I can see only one reason, the same for both (8) and (9). In both cases (a) has a certain human colouring, by suggesting that part of the mathematics is carried out by a human being. This adds nothing to the mathematical content, but somehow it helps the readability.

Mathematical writers know they have to be careful about adding human content. Anything that distracts from the argument will offend some readers. (When I first suggested the names Abelard and Eloise for the two players in a logical game, I had some stick from fellow logicians who disliked having the games personalised. I had suggested the names in order to make the pronouns 'he' and 'she' available to name the players, more for convenience than for human colour.) In fact there seem to be a fairly small and stereotyped set of personalisations that turn up regularly in the literature.

One is well known: we say 'We can find an X' instead of 'There is an X'. This device is not a discovery of the mathematicians—a version of it is built into several major languages. The French say *il se trouve*, the Russians *nakhoditsya* and the Arabs *yūjadu*; all these phrases express 'there exists' by saying 'there is found'.

The examples in (53.8) and (67.22) both illustrate a different idiom. The human isn't finding something; instead he or she is entering into the mathematics by *performing a function*. In spite of the attempts of set theorists to persuade us to think of functions as sets of ordered pairs, we persist in thinking of them as things that a person can do (i.e. has the power of doing). With any function $f(x, y)$ we think of someone taking the arguments (a, b) and turning them into the value $f(a, b)$. Actually the metaphor in (8) and (9) is a more specialised one that applies to embeddings and similar functions: we imagine somebody taking the structure A and planting it inside the structure B that it is embedded into—just as one plants a bush in the garden.

There is a cost in metaphors of this kind. Generally what they say isn't literally true. We can't cause a mathematical structure A to be embedded in another structure B; in general the most we can do is to describe an embedding. But very often we can't even do that, even when an embedding exists; it would take more than a lifetime to write out the description, or to compute what it is. Some embeddings can't be defined at all with the notions available to us. So if these metaphors were taken literally, they would imply we have magical powers.

I don't believe any working mathematicians are taken in by these exaggerations. In fact I suspect most mathematicians never consciously notice them. But there is a difficulty for historians of mathematics. Take for example Postulate 1 from Euclid's Elements:

> To draw a straight line from any point to any point.

Today one of the standard axioms of the affine plane says that through any two distinct points there is a unique straight line. So if a modern geometer says 'Given the two points a and b, draw the straight line through them', we say the metaphor is harmless because it can be replaced by a reference to the axiom. But Euclid didn't have axioms for the affine plane. He didn't have a formal system that would have allowed him to paraphrase Postulate 1 without the reference to the human activity of drawing.

So one is tempted to say—and I have certainly heard people say it—that Euclid must have meant Postulate 1 literally. For example one might say that Euclid's geometry is about activities that a human being can perform, like drawing a line, in contrast to modern geometry which is not about human activities at all. Possibly this is a correct description of Euclid's geometry; but one moral of this paper is that you can't infer such a thing from phrases like his Postulate 1. Modern mathematical writing is deeply coloured with metaphors about human powers and activities, and maybe the same was already true in Euclid's time.

Euclid himself doesn't discuss whether Postulate 1 should be taken literally—any more than Birkhoff and Mac Lane discuss how literally we should take their phrases about 'can embed'. But some ancient philosophers of mathematics did. The Platonist philosopher Simplicius (6th century AD) wrote a commentary on Book I of Euclid's Elements, in which he made the following comment on Postulate 1:

(10) As for this postulate, it is necessary to ask that it be postulated because the existence of geometrical matter is in the imagination. For, indeed, if their essences were in material bodies, it would be rash to postulate that a straight line be drawn from Aries to Libra. ([5] p. 92)

This carries the clear implication that Euclid's Postulate should not be read as describing what we can literally 'draw'.

3 'must'

Of the 340 items in BM, 87 (=26%) would normally be classified as $\Box$ modalities ('necessary', 'must', 'have to' etc.). Many of these 87 examples are just negations of the corresponding $\Diamond$ modalities, for example:

(17.58) The set S **can** contain **nothing but** the integral multiples of b.
(41.28) This is **not** strictly **possible**.
(46.1) ... the equations are trivial; they **can** have **no** solution unless ...
(67.40) ... **can** be expressed in one and **only one** way as a polynomial form ....

But 33 of the 87 contain the word 'must'.

I want to concentrate on a group of seven examples which take the form

Since (or if) A holds, B must hold.

or some close variant of this form. Here are four of them:

(19.1) Since the remainders continually decrease, there **must** ultimately be a remainder r_{n+1} which is zero.
(46.20) ... any set of elements $x_1, \ldots, x_n$ in F which satisfies (E′) **must** satisfy (E).
(74.38) $x^2 \equiv 2$ (mod 5) is **impossible** for $x \in J$.
(90.2) Two real numbers $(a : b)$ and $(c : d)$ **can** be different **only** when there is a rational number greater than one and smaller than the other.

This idiom is used when there is a correct piece of reasoning that infers B from A—in other words, when we **can** deduce B from A. Note for future reference the startling switch of modality: 'must' should be a $\Box$ modality but 'can' is a $\Diamond$ one.

In my youth I was taught the classical view, that the sentence

(11) If A holds then B must hold.

should normally be read as meaning

(12) Necessarily: If A holds then B holds.

We used to be shown examples of fallacious arguments where people failed to realise that (11) means (12), so that they finished up deducing the necessary truth of some contingent statement.

There certainly are conditionals with a modal conclusion, where the modal word has to be understood as applying to the whole sentence. Palmer [7] p. 185 gives the example

(13) If John came, Mary **intended** to leave.

(You can make it feel more modal by putting 'was going to leave' in place of 'intended to leave'.) The meaning of (13) is

(14) Mary had the intention that if John came, she would leave.

I give this example to contrast it with the mathematical ones.

Take for example (19.1). Pushing back the necessity turns (19.1) into something like:

(15) It's necessarily the case that since the remainders continually decrease, there is ultimately a remainder r_{n+1} which is zero.

This is not quite right, because (19.1) never suggested that there is anything necessary about the remainders decreasing. I think we can cover that point by switching 'since' to 'if' in both (19.1) and (15):

(16) It's necessarily the case that if the remainders continually decrease, there is ultimately a remainder r_{n+1} which is zero.

If the mathematics in (19.1) is correct, then no doubt (16) is true. But I have severe doubts that (16) represents the force of 'must' in (19.1). The basic objection is the point at the beginning of this paper: the necessity of the conditional in (16) is irrelevant to the argument. Mathematical authors wouldn't want to distract the reader by even mentioning this necessity.

For the same reason, we shouldn't read the 'must' in (19.1) itself as expressing that it's a necessary truth that there is a zero remainder. Even if it was a necessary truth, this would be irrelevant to the argument.

So why do Birkhoff and Mac Lane include that word 'must'? One route to an answer is to try leaving it out. Compare the following, where (a) is (19.1) and (b), (c) are paraphrases of it:

(17) (a) Since the remainders continually decrease, there must ultimately be a remainder r_{n+1} which is zero.
(b) Since the remainders continually decrease, there is ultimately a remainder r_{n+1} which is zero.
(c) The remainders continually decrease, so there must ultimately be a remainder r_{n+1} which is zero.

What does (a) give us that (b) and (c) didn't?

As far as the mathematical argument goes, there is nothing to choose between (a), (b) and (c). To me they are stylistic variants. The reason for the word 'since' in (b) is the same as the reason for 'so there must be' in (c), and in (a) the same job is shared between 'since' and 'there must be'. In each case the job of these words is to structure the argument by telling the reader that the consequent is being derived from the antecedent. These words are not part of the mathematical argument. They are *formatting* words, that guide the reader along the structure of the argument.

In short, the role of 'must' in the consequent of (19.1) is to indicate to the reader 'You should read this clause as justified by a clause nearby'. To do that job, the word has to be where it is, in the consequent and not at the front of the whole conditional.

Now I ask a more speculative question: Why is the word 'must' used in this role? Maybe I can throw some small light on this—though everything I can offer may be overruled by the historians of language.

In English the main uses of the word 'must' fall into two groups, the epistemic as in

(18) You must find it quite a change being back in London.

and the deontic as in

(19) The University is saying 'These people must be expelled if they disrupt lectures'.

(Example (18) is from Palmer [7] p. 53, and (19) from p. 73 in the same book.) In my talk in Kolkata I suggested that 'must' in (19.1) should be read as deontic, and I speculated that there is a metaphor in the background: we think of A as withholding permission from B not to hold. I still think that there is a metaphor of permission, but I think I got it the wrong way round.

The force of 'must' in (18) is that the speaker infers 'You are finding it quite a change' from the information that the addressee has moved back to London. This is very close to the mathematical usage, with the trivial change that the basis of the inference comes after the fact inferred, not immediately before. (One could think up mathematical examples with the basis after the fact inferred.) So I now take it that 'must' in (19.1) and its fellows above is an epistemic 'must'.

The curious thing is that the Old English ancestor *mōtan* of the word 'must', together with its German counterpart, actually meant 'be allowed to'. Could it be that 'you are allowed to X' shaded into 'I am allowed to believe that you X'? In that case the word 'must', attached to a consequent as in (19.1), is actually a $\Diamond$ modality qualifying the consequent. Its force is 'At this stage in the discussion we are allowed to assert that …'. (Bear in mind that in mathematics, if we are allowed to assert p and we are allowed to assert q, then we are allowed to assert 'p and q'.)

If this interpretation of 'must' in mathematical English is broadly correct, then another group of phrases falls into place beside it. A common idiom in BM is 'we can prove', 'we may deduce' etc., as for example in

> (24.8) …from this we **may** derive the conclusions in the form $m|(a+x-b-x)\ldots.$

There are 51 such items (=15%), of which 22 have the verb 'may'.

'May' is not normally used with powers of organisms:

> ??Plants with chlorophyll may extract energy from sunlight. (meaning 'can')

A much commoner use of 'may' is to express permission. This suggests a natural metaphor behind (24.8): 'this' (some fact previously established) gives us permission to write $m|(a+x-b-x)$ as the next step in the reasoning. If this is what is going on, then 'may' and 'must' play very similar roles, though their syntax is a little different. The counterpart of (24.8) using 'must' would be

(20) …from this we must have $m|(a+x-b-x)$.

So 'may' falls into place as another formatting word alongside 'must'.

Three of the examples in BM do refer explicitly to permission:

> (13.10) The Principle of Finite Induction **permits** one to assume the truth of $P(n)$ *gratis* in proving $P(n+1)$.
> (24.21) Theorem 10 **allows** us to conclude that ...
> (62.footnote) ... every manipulation **allowed** on x must be true for every possible value of x.

Of these, (24.21) is the only one which says that we are allowed to make a deduction. But in a style of mathematics where assumptions are made and discharged, we should expect some formatting words that call attention to the places where these things happen, as in (13.10).

4 Conclusions

- Modal notions appear frequently in mathematical writing.
- Generally they express metaphors, e.g. about powers of organisms, or about permissions. (There are other metaphors we haven't discussed.)
- Two main uses of modal words in mathematical writing are for *colouring* to help readability, and for *formatting* to guide the reader through the structure of the reasoning.
- There is no need to invoke any notion of 'mathematical necessity' or 'logical necessity' to explain these usages.

There is a particular conclusion for philosophers who read papers by mathematical logicians. Mathematical logicians do sometimes write papers about mathematical properties of modal notions. The obvious example is modal logic treated mathematically. We must expect that papers in this area include modal notions in two different ways: first as the mathematical subject matter, and second as colour-and-format. Both reader and writer have a duty to make sure that these two uses of modality don't get confused. In practice there is usually no problem, because the modal subject-matter is expressed with symbols like $\Box$ and $\Diamond$, whereas the format and colouring are done in English.

The danger area is where mathematical logicians write in non-technical English for philosophers, particularly if there is some doubt about whether they really are discussing modal notions. Alfred Tarski has suffered badly here. Though a mathematician by training and inclination, he wrote a number of papers for philosophers. In one of them he modestly described himself as

> a mathematician (as well as a logician, and perhaps a philosopher of a sort) ([9] p. 369).

None of his published papers were about modal logic. In fact Tarski seems to have distanced himself from modal logic. In 1946 he dismissed the Lewis modal systems and other 'many-valued' systems with the remark

> ...I should say that the only one of these systems for which there is any hope of survival is [the quantum logic] of Birkhoff and von Neumann. ([8] p. 25; see Sinaceur's note 25 on page 31 for Tarski's reason for including modal logics among many-valued logics.)

He is said to have disowned any connection between his work with Jońsson on boolean operators and its later rediscovery in the setting of Kripke structures for modal logics. So if modal notions appear in Tarski's papers, the default assumption should be that they are there for format or colouring, not as part of the argument.

A few years back there was a debate among North American philosophers about whether Tarski in his papers on logical consequence handled the notion of logical necessity correctly. Someone should have checked first whether the relevant papers of Tarski ever mentioned logical necessity at all. The passages in question seem to me typical examples of the 'formatting' that we discussed in the previous section.

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